

A generalization of the Hyers–Ulam–Rassias stability of the beta functional equation

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Abstract. In this paper, we prove a generalization of the Hyers–Ulam–Rassias stability for the inverse form (2') of the beta functional equation. As a consequence we obtain the Hyers–Ulam stability and the stability in the spirit of Găvruta for the gamma functional equation.

1. Introduction

In 1940, S. M. ULAM [16] raised the following problem: Under what conditions does there exist an additive mapping near an approximately additive mapping?

In 1941, this problem was solved by D. H. HYERS [3]. Thereafter we usually say that the equation $E_1(h) = E_2(h)$ is stable in the Hyers–Ulam sense if for an approximate solution f of this equation, i.e. for a function f with $|E_1(f) - E_2(f)| \leq \delta$ there exists a function g such that $E_1(g) = E_2(g)$ and $|f(x) - g(x)| \leq \epsilon$. In 1978 the Hyers–Ulam stability for additive mapping was generalized by TH. M. RASSIAS [12]. This result of TH. M. RASSIAS was again generalized by P. GĂVRUTA [2] as follows:

If for an approximate solution f of the equation $E_1(h) = E_2(h)$, i.e. for a function f such that $|E_1(f) - E_2(f)| \leq \phi$ holds with a given function ϕ there exists a function g such that $E_1(g) = E_2(g)$ and $|g(x) - f(x)| \leq \Phi(x)$ for some fixed function Φ .

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The functional equation

$$(1) \quad f(x+1) = xf(x) \quad \text{for all } x > 0$$

is called the gamma functional equation. It is well-known that the gamma function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \quad (x > 0)$$

is a solution of the gamma functional equation.

From the relation of gamma and beta function, that is,

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(y, x),$$

the functional equation

$$(2) \quad f(x+1, y+1) = \frac{xy}{(x+y)(x+y+1)} f(x, y) \quad \text{for all } x, y > 0$$

will be called the beta functional equation. The beta function

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$$

is a solution of the beta functional equation.

We consider the inverse functional equation of beta functional equation (2) as follows:

$$(2') \quad B(x+1, y+1)^{-1} = \frac{(x+y)(x+y+1)}{xy} B(x, y)^{-1}.$$

In this paper, we shall investigate the modified Hyers–Ulam–Rassias stability of the functional equation (2'). Throughout this paper, we denote by $\mathbb{R}_+$ the set of all positive real numbers and n_0 is a nonnegative integer, and in particular the author will use a notation $x_i = x + i$ for the convenience of calculation and the appreciation of the reader. By using an idea of GĂVRUTA [2] we can prove the following theorem:

Theorem 1. *Let $\Phi : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a given mapping satisfying the inequality*

$$(3) \quad \Phi(x, y) := \sum_{j=0}^{\infty} \varphi(x_j, y_j) \prod_{i=0}^j \frac{x_i y_i}{(x_i + y_i)(x_i + y_i + 1)} < \infty$$

for all $x, y \in \mathbb{R}_+$, and let n_0 be a given nonnegative integer.

Assume that a mapping $B : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies the inequality

$$(4) \quad \left| B(x+1, y+1)^{-1} - \frac{(x+y)(x+y+1)}{xy} B(x, y)^{-1} \right| \leq \varphi(x, y)$$

for all $x, y > n_0$. Then there exists a unique mapping $T : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which satisfies the beta functional equation (2') and the inequality

$$(5) \quad |T(x, y)^{-1} - B(x, y)^{-1}| \leq \Phi(x, y)$$

for all $x, y > n_0$.

2. Proof of the Theorem 1

For $x, y > n_0$, we use an induction on n to prove

$$(6) \quad \left| B(x_n, y_n)^{-1} - B(x, y)^{-1} \prod_{i=0}^{n-1} \frac{(x_i + y_i)(x_i + y_i + 1)}{x_i y_i} \right| \\ \leq \sum_{j=0}^{n-1} \varphi(x_j, y_j) \prod_{i=1}^{n-1-j} \frac{(x_{i+j} + y_{i+j})(x_{i+j} + y_{i+j} + 1)}{x_{i+j} y_{i+j}}$$

for all positive integers n , where we assume that $\prod_{i=1}^0 \frac{(x_i + y_i)(x_i + y_i + 1)}{x_i y_i} = 1$ conventionally. The inequality (6) immediately follows from (4) for the case of $n = 1$. If we assume that (6) holds true for some n , then we obtain for $n + 1$

$$\left| B(x_{n+1}, y_{n+1})^{-1} - B(x, y)^{-1} \prod_{i=0}^n \frac{(x_i + y_i)(x_i + y_i + 1)}{x_i y_i} \right| \\ \leq \left| B(x_{n+1}, y_{n+1})^{-1} - \frac{(x_n + y_n)(x_n + y_n + 1)}{x_n y_n} B(x_n, y_n)^{-1} \right| \\ + \frac{(x_n + y_n)(x_n + y_n + 1)}{x_n y_n} \\ \cdot \left| B(x_n, y_n)^{-1} - B(x, y)^{-1} \prod_{i=0}^{n-1} \frac{(x_i + y_i)(x_i + y_i + 1)}{x_i y_i} \right| \\ \leq \varphi(x_n, y_n) + \frac{(x_n + y_n)(x_n + y_n + 1)}{x_n y_n}$$

$$\begin{aligned}
& \cdot \sum_{j=0}^{n-1} \varphi(x_j, y_j) \prod_{i=1}^{n-1-j} \frac{(x_{i+j} + y_{i+j})(x_{i+j} + y_{i+j} + 1)}{x_{i+j} y_{i+j}} \\
& = \varphi(x_n, y_n) + \sum_{j=0}^{n-1} \varphi(x_j, y_j) \prod_{i=1}^{n-j} \frac{(x_{i+j} + y_{i+j})(x_{i+j} + y_{i+j} + 1)}{x_{i+j} y_{i+j}} \\
& = \sum_{j=0}^n \varphi(x_j, y_j) \prod_{i=1}^{n-j} \frac{(x_{i+j} + y_{i+j})(x_{i+j} + y_{i+j} + 1)}{x_{i+j} y_{i+j}},
\end{aligned}$$

which completes the proof of (6). If we divide both sides in (6) by $\prod_{i=0}^{n-1} \frac{(x_i + y_i)(x_i + y_i + 1)}{x_i y_i}$, then we get

$$\begin{aligned}
(7) \quad & \left| B(x_n, y_n)^{-1} \prod_{i=0}^{n-1} \left(\frac{(x_i + y_i)(x_i + y_i + 1)}{x_i y_i} \right)^{-1} - B(x, y)^{-1} \right| \\
& \leq \sum_{j=0}^{n-1} \varphi(x_j, y_j) \prod_{i=0}^j \left(\frac{(x_i + y_i)(x_i + y_i + 1)}{x_i y_i} \right)^{-1}
\end{aligned}$$

for every $n \in \mathbb{N}$. By using (4) and (3) we have for $n > m > 0$

$$\begin{aligned}
& \left| B(x_m, y_m)^{-1} \prod_{i=0}^{m-1} \left(\frac{(x_i + y_i)(x_i + y_i + 1)}{x_i y_i} \right)^{-1} \right. \\
& \quad \left. - B(x_n, y_n)^{-1} \prod_{i=0}^{n-1} \left(\frac{(x_i + y_i)(x_i + y_i + 1)}{x_i y_i} \right)^{-1} \right| \\
& = \left| B(x_m, y_m)^{-1} \prod_{i=0}^{m-1} \left(\frac{(x_i + y_i)(x_i + y_i + 1)}{x_i y_i} \right)^{-1} \right. \\
& \quad - B(x_{m+1}, y_{m+1})^{-1} \prod_{i=0}^m \left(\frac{(x_i + y_i)(x_i + y_i + 1)}{x_i y_i} \right)^{-1} \\
& \quad + \dots \\
& \quad + B(x_{n-1}, y_{n-1})^{-1} \prod_{i=0}^{n-2} \left(\frac{(x_i + y_i)(x_i + y_i + 1)}{x_i y_i} \right)^{-1} \\
& \quad \left. - B(x_n, y_n)^{-1} \prod_{i=0}^{n-1} \left(\frac{(x_i + y_i)(x_i + y_i + 1)}{x_i y_i} \right)^{-1} \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=m}^{n-1} \left| B(x_j, y_j)^{-1} \frac{(x_j + y_j)(x_j + y_j + 1)}{x_j y_j} \right. \\
&\quad \left. - B(x_{j+1}, y_{j+1})^{-1} \prod_{i=0}^j \left(\frac{(x_i + y_i)(x_i + y_i + 1)}{x_i y_i} \right)^{-1} \right| \\
&\leq \sum_{j=m}^{n-1} \varphi(x_j, y_j) \prod_{i=0}^j \left(\frac{(x_i + y_i)(x_i + y_i + 1)}{x_i y_i} \right)^{-1} \rightarrow 0, \\
&\text{as } m \rightarrow \infty.
\end{aligned}$$

Therefore, the sequence

$$B(x_n, y_n)^{-1} \prod_{i=0}^{n-1} \left(\frac{(x_i + y_i)(x_i + y_i + 1)}{x_i y_i} \right)^{-1}$$

is a Cauchy sequence for $x, y > n_0$, and hence we can define a mapping $T_0 : (n_0, \infty) \times (n_0, \infty) \rightarrow \mathbb{R}_+$ by

$$(8) \quad T_0(x, y)^{-1} = \lim_{n \rightarrow \infty} B(x_n, y_n)^{-1} \prod_{i=0}^{n-1} \left(\frac{(x_i + y_i)(x_i + y_i + 1)}{x_i y_i} \right)^{-1}$$

for all $x, y > n_0$. Letting in (7) $n \rightarrow \infty$ and applying (8) and (3) we obtain (5).

By (8) we can easily verify that T_0 satisfies (2'):

$$\begin{aligned}
&T_0(x+1, y+1)^{-1} \\
&= \lim_{n \rightarrow \infty} B(x_{n+1}, y_{n+1})^{-1} \prod_{i=0}^{n-1} \left(\frac{(x_{i+1} + y_{i+1})(x_{i+1} + y_{i+1} + 1)}{x_{i+1} y_{i+1}} \right)^{-1} \\
&= \frac{(x+y)(x+y+1)}{xy} \lim_{n \rightarrow \infty} B(x_{n+1}, y_{n+1})^{-1} \prod_{i=0}^n \left(\frac{(x_i + y_i)(x_i + y_i + 1)}{x_i y_i} \right)^{-1} \\
&= \frac{(x+y)(x+y+1)}{xy} T_0(x, y)^{-1}
\end{aligned}$$

for all $x, y > n_0$.

Now we assume that $G : (n_0, \infty) \times (n_0, \infty) \rightarrow \mathbb{R}_+$ is another mapping which satisfies (2') as well as (5) for all $x, y > n_0$. By (2'), (5) and (3) we

obtain

$$\begin{aligned}
& |T_0(x, y)^{-1} - G(x, y)^{-1}| \\
&= |T_0(x_n, y_n)^{-1} - G(x_n, y_n)^{-1}| \prod_{i=0}^{n-1} \frac{x_i y_i}{(x_i + y_i)(x_i + y_i + 1)} \\
&\leq 2\Phi(x_n, y_n) \prod_{i=0}^{n-1} \frac{x_i y_i}{(x_i + y_i)(x_i + y_i + 1)} \\
&= 2 \sum_{j=0}^{\infty} \varphi(x_{n+j}, y_{n+j}) \prod_{i=0}^{n+j} \frac{x_i y_i}{(x_i + y_i)(x_i + y_i + 1)} \\
&= 2 \sum_{j=n}^{\infty} \varphi(x_j, y_j) \prod_{i=0}^j \frac{x_i y_i}{(x_i + y_i)(x_i + y_i + 1)} \rightarrow 0, \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

for all $x, y > n_0$. This implies the uniqueness of T_0 . Now we extend the function T_0 to $(0, \infty) \times (0, \infty)$. We define for $0 < x, y \leq n_0$

$$T(x, y) := \prod_{n=0}^{k-1} \frac{(x_n + y_n)(x_n + y_n + 1)}{x_n y_n} \cdot T_0(x + k, y + k),$$

where k is the smallest natural number satisfying the inequalities $x + k > n_0$ and $y + k > n_0$. And also $T(x, y) = T_0(x, y)$ for all $x, y > n_0$. Then $T(x + 1, y + 1) = \frac{xy}{(x+y)(x+y+1)} T(x, y)$ for all $x, y > 0$.

Also the following inequality holds

$$|T(x, y)^{-1} - B(x, y)^{-1}| < \Phi(x, y)$$

for all $x, y > n_0$. Hence, the proof of the theorem is completed.

3. Applications to the gamma and beta functional equation

The following corollary that is called the Hyers–Ulam stability for the functional equation (2') can be found in the author's papers ([7], [11]).

Corollary 2. Assume that a mapping $B : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies the inequality

$$\left| B(x + 1, y + 1)^{-1} - \frac{(x + y)(x + y + 1)}{xy} B(x, y)^{-1} \right| \leq \delta$$

for some $\delta > 0$ and for all $x, y > n_0$. Then there exists a unique mapping $T : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which satisfies the beta functional equation (2') and the inequality

$$|T(x, y)^{-1} - B(x, y)^{-1}| \leq \delta$$

for all $x, y > n_0$.

PROOF. Apply Theorem 1 and condition (3) with $\varphi(x, y) = \delta$. Then we arrive

$$\begin{aligned} \Phi(x, y) &= \sum_{j=0}^{\infty} \delta \prod_{i=0}^j \frac{x_i y_i}{(x_i + y_i)(x_i + y_i + 1)} = \delta \sum_{j=0}^{\infty} \prod_{i=0}^j \frac{x_i y_i}{(x_i + y_i)(x_i + y_i + 1)} \\ &< \delta \left(\frac{1}{2} + \frac{1}{2^2} + \cdots \right) = \delta. \end{aligned} \quad \square$$

For the stability of the gamma functional equation we apply Theorem 1 to a single variable, and then we can get the following results. In the case $n_0 = 0$, Corollary 4 can be found in S.-M. JUNG ([8], [9]), H. ALZER [1] and the author's [10].

Theorem 3. Assume that a mapping $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies the inequality

$$(9) \quad |g(x+1) - xg(x)| \leq \varphi(x)$$

for all $x, y > n_0$. Then there exists a unique mapping $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which satisfies the gamma functional equation (1) with

$$|f(x) - g(x)| \leq \Phi(x) \quad \forall x > n_0,$$

where $\Phi(x) := \sum_{j=0}^{\infty} \varphi(x_j) \prod_{i=0}^j \frac{1}{x_i} < \infty$.

PROOF. For any $x > n_0$ and for every positive integer n we define

$$P_n(x) = g(x_n) \prod_{i=0}^{n-1} \frac{1}{x_i}.$$

By (9) we have

$$\begin{aligned} (10) \quad |P_{n+1}(x) - P_n(x)| &= |g(x_{n+1}) - x_n g(x_n)| \prod_{i=0}^n \frac{1}{x_i} \\ &\leq \varphi(x_n) \prod_{i=0}^n \frac{1}{x_i} \quad \text{for } x > n_0. \end{aligned}$$

Now we use induction on n to prove

$$(11) \quad |P_n(x) - g(x)| \leq \sum_{j=0}^{n-1} \varphi(x_j) \prod_{i=0}^j \frac{1}{x_i}$$

for the fixed $x > n_0$ and for all positive integers n . For the case $n = 1$, the inequality (11) is an immediate consequence of (9). Assume that (11) holds true for some n . It then follows from (9) and (10)

$$|P_{n+1}(x) - g(x)| \leq |P_{n+1}(x) - P_n(x)| + |P_n(x) - g(x)| \leq \sum_{j=0}^n \varphi(x_j) \prod_{i=0}^j \frac{1}{x_i}.$$

which completes the proof of (11). Now let m, n be positive integers with $n \geq m$. Suppose $x(> n_0)$ is given. By definition of Φ , we have

$$\begin{aligned} |P_n(x) - P_m(x)| &\leq \sum_{j=m}^{n-1} |P_{j+1}(x) - P_j(x)| \\ &\leq \sum_{j=m}^{n-1} \varphi(x_j) \prod_{i=0}^j \frac{1}{x_i} \rightarrow 0 \quad \text{as } m \rightarrow \infty. \end{aligned} \quad \square$$

This implies that $\{P_n(x)\}$ is a Cauchy sequence for $x > n_0$. Next proceeding of the proof is very similiar to that of the Theorem 1.

Corollary 4. Assume that a mapping $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies the inequality

$$|g(x+1) - xg(x)| \leq \delta$$

for some $\delta > 0$ and for all $x, y > n_0$. Then there exists a unique mapping $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which satisfies the gamma functional equation (1) with

$$|f(x) - g(x)| \leq \frac{e\delta}{x}$$

for all $x > n_0$, where e is the best possible constant.

PROOF. Apply $\delta = \varphi(x)$ in Theorem 3. We can find in [1] that e is the best possible constant. $\square$

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