

A characterization of the Janko group J_1 by an active fragment of its character table

By V. A. BELONOGOV (Ekaterinburg)

Abstract. The sporadic simple Janko group J_1 of order 175560 has a 15×15 character table X . In this paper it is given a characterization of J_1 by a 12×6 submatrix of X , which is an active fragment of X . The notion of the active fragment introduced in the author's book "Representations and characters in the theory of finite groups", Sverdlovsk, 1990 (in Russian).

1. Introduction

Let G be a finite group, X a character table of G , D the union of some conjugacy classes of G (a normal subset of G) and Φ a set of irreducible characters of G . Denote by $X(\Phi, D)$ the submatrix of X consisting of the values of the functions from Φ on the elements from D . Let us say that $X(\Phi, D)$ is an *active fragment* of X if D and Φ interact [1] (see Section 2 below).

The sporadic simple group J_1 of order 175560 has a 15×15 character table X [5]. It is easy to see that X has the following 12×6 active fragment:

$$A = \begin{array}{l} \\ \\ \\ \\ \end{array} \begin{array}{c} 3 \\ \downarrow \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 56 & 0 & 1 & -1 \\ 76 & -1 & -1 & 0 \\ 77 & 0 & 0 & 1 \\ 133 & 0 & 1 & 0 \\ 209 & -1 & 0 & 0 \end{bmatrix} \end{array}.$$

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Here the matrix A is written in a compact form: a number with an arrow indicates the multiplicity of the corresponding row or column. (The full form of A is written in Lemma 2 below.)

The aim of this paper is to prove

Theorem. *Let G be a finite group. The following conditions are equivalent:*

- (1) *the matrix A as above is an active fragment of a character table of G ,*
- (2) *$G \cong J_1$.*

Note that similar characterizations of the groups $PSL(2, q)$ and $Sz(q)$ are obtained in [2, Chapter 8], where some general methods of investigation of a group with a given active fragment of its character table are developed.

Let G be a group satisfying the condition (1). The proof of Theorem, as proofs of the characterizations above, includes the following steps to make clear the structure of G : the determination of the order of G , the extension of A to a larger fragment of a character table of G , the proof of simplicity of G . Besides the proof of Theorem contains a new step: the complete reconstruction of the centralizer of an involution of G . Finally, a result of JANKO [5] completes the proof of Theorem.

2. Preliminaries

We use the standard notation and some definitions from [1], [2]. Let us recall something of these. Always G is a finite group, $\text{Irr}(G)$ is the set of irreducible (complex) characters of G , $g^G := \{x^{-1}gx \mid x \in G\}$ and $o(g)$ is the order of g for $g \in G$, $k(G)$ is the number of the conjugacy classes of G , $k_G(D)$ is the number of the conjugacy classes of G contained in a normal subset D of G , $M \times N$ is the Cartesian product of sets M and N . If α is a class function of G and D is a normal subset of G then $\alpha|_D^0$ is a class function of G defined by the equation

$$\alpha|_D^0(g) = \begin{cases} 0 & \text{if } g \in G \setminus D, \\ \alpha(g) & \text{if } g \in D. \end{cases}$$

The function $\alpha|_D^0$ is called the D -part of α .

Suppose that D is a normal subset of G and $\Phi \subseteq \text{Irr}(G)$. We say that D and Φ *interact* if the D -part $\phi|_D^0$ of every character ϕ in Φ is a linear combination of the characters in Φ : $\phi|_D^0 = \sum_{\psi \in \Phi} a_{\phi, \psi} \psi$, $a_{\phi, \psi}$ are complex numbers. (It is shown in [1], [2] that the classical concept of p -block of irreducible characters of a group may be defined in terms of interactions; note that every p -section of G interacts with every p -block of G .) We set

$$D^- = G \setminus D, \quad \Phi^- = \text{Irr}(G) \setminus \Phi.$$

Proposition 1. *Let G be a finite group, D a normal subset of G and $\Phi \subseteq \text{Irr}(G)$. The following conditions are equivalent:*

- (1) D and Φ interact,
- (2) $\sum_{d \in D} \phi(d) \overline{\chi(d)} = 0$ for all $(\phi, \chi) \in \Phi \times \Phi^-$,
- (3) $\sum_{\phi \in \Phi} \phi(d) \overline{\phi(g)} = 0$ for all $(d, g) \in D \times D^-$,
- (4) $|G|^{-1} \sum_{d \in D} \sum_{\phi \in \Phi} \psi(d) \overline{\phi(d)} \phi(t) = \psi(t)$ for all $(\psi, t) \in \Phi \times D$.

PROOF. See [1, Theorem 1]. □

Proposition 2. *Let X be a character table of a group G , D is a normal subset of G and $\Phi \subseteq \text{Irr}(G)$. The following conditions are equivalent:*

- (1) $\text{rank}(X(\Phi, D)) + \text{rank}(X(\Phi, D^-)) = |\Phi|$,
- (2) $\text{rank}(X(\Phi, D)) + \text{rank}(X(\Phi^-, D)) = k_G(D)$,
- (3) $\text{rank}(X(\Phi, D)) + \text{rank}(X(\Phi, D^-)) + \text{rank}(X(\Phi^-, D)) + \text{rank}(X(\Phi^-, D^-)) = k(G)$,
- (4) D and Φ interact.

PROOF. See [2, Theorem 8A6]. □

3. Proof of Theorem

Troughout this section, G is a finite group which satisfies the condition (1) of Theorem. Let a, b, c, d_1, d_2, d_3 be elements of G , corresponding to the first, second, $\dots$, sixth column of A , respectively, and ϕ_i ($i = 1, \dots, 12$)

denotes the element of $\text{Irr}(G)$, corresponding to i -th row of A . Let us set

$$D = a^G \cup b^G \cup c^G \cup d_1^G \cup d_2^G \cup d_3^G \quad \text{and} \quad \Phi = \{\phi_1, \dots, \phi_{12}\}.$$

Thus, $A = X(\Phi, D)$ for a character table X of G .

Lemma 1. (1) $|G| = 175560 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$.

(2) $|C_G(b)| = 7$, $|C_G(c)| = 11$.

(3) $|C_G(d_1)|^{-1} + |C_G(d_2)|^{-1} + |C_G(d_3)|^{-1} = 3/19$.

PROOF. Let x^{-1} , y^{-1} , z^{-1} , t_1^{-1} , t_2^{-1} , t_3^{-1} denote the orders of the centralizers of a , b , c , d_1 , d_2 , d_3 , respectively. By Proposition 1 ((1) $\implies$ (4)), we have the equation

$$A \text{diag}(x, y, z, t_1, t_2, t_3) A^* A = A.$$

Since $\text{rank}(A) = 3$ and the 3×3 submatrix $A_{(2,3,4)}^{(6,9,12)}$ of A (superscripts are numbers of rows, subscripts are numbers of column) is non-singular, the equation is equivalent to

$$A^{(6,9,12)} \text{diag}(x, y, z, t_1, t_2, t_3) A^* A_{(2,3,4)} = A_{(2,3,4)}^{(6,9,12)},$$

that is

$$\begin{bmatrix} 77x & 0 & 0 & t_1 & t_2 & t_3 \\ 133x & 0 & z & 0 & 0 & 0 \\ 209x & -y & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -360 & 360 & 120 \\ 4 & 3 & 1 \\ 3 & 8 & -1 \\ 1 & -1 & 6 \\ 1 & -1 & 6 \\ 1 & -1 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

From here we get $x = 1/175560$, $y = 1/7$, $z = 1/11$, $t_1 + t_2 + t_3 = 3/19$. Therefore the assertions (2) and (3) hold.

Now we show that $a=1$. If not, $1 \in D^-$ and Proposition 1 ((1) $\implies$ (3) for $(d, g) = (a, 1)$) implies a contradiction. Hence $a = 1$, $|G| = x^{-1}$ and (1) holds. $\square$

Lemma 2. (1) $k(G) = 15$ and the active fragment A may be extended to the following fragment of a character table of G :

$ C_G(g) $ g	$ G $ 1	7 b	11 c	19 d_1	19 d_2	19 d_3	$g_1 \dots g_9$
ϕ_1	1	1	1	1	1	1	1 ... 1
ϕ_2	56	0	1	-1	-1	-1	
ϕ_3	56	0	1	-1	-1	-1	
ϕ_4	76	-1	-1	0	0	0	
ϕ_5	76	-1	-1	0	0	0	
ϕ_6	77	0	0	1	1	1	
ϕ_7	77	0	0	1	1	1	
ϕ_8	77	0	0	1	1	1	
ϕ_9	133	0	1	0	0	0	
ϕ_{10}	133	0	1	0	0	0	
ϕ_{11}	133	0	1	0	0	0	
ϕ_{12}	209	-1	0	0	0	0	
χ_1	120	1	-1	a_{11}	a_{12}	a_{13}	0 ... 0
χ_2	120	1	-1	a_{21}	a_{22}	a_{23}	0 ... 0
χ_3	120	1	-1	a_{31}	a_{32}	a_{33}	0 ... 0

(here a_{ij} are some complex numbers);

(2) $\sum_{i=1}^3 a_{ij} = -1$ for all $j \in \{1, 2, 3\}$;

(3) $|C_G(g_i)|$ divides $8 \cdot 3 \cdot 5$ for all $i \in \{1, \dots, 9\}$.

PROOF. Since $\text{rank}(X(\Phi, D)) = 3$ and $k_G(D) = 6$, $\text{rank}(X(\Phi^-, D)) = 6 - 3 = 3$ by Proposition 2. Therefore, $|\Phi^-| \geq 3$. By Proposition 1 ((1) $\implies$ (2)), we have $\phi_{12}(1)\overline{\chi(1)} - |b^G|\phi_{12}(b)\overline{\chi(b)} = 0$ for $\chi \in \Phi^-$, whence $\chi(1) = 120\chi(b)$ and $\chi(b)$ is integer (for example, by [2, 2A6(3)]). Hence

$$\sum_{\chi \in \Phi^-} \chi(1)^2 = 120^2 \sum_{\chi \in \Phi^-} \chi(b)^2 \geq 120^2 |\Phi^-| \geq 120^2 \cdot 3.$$

But by the second orthogonality relation, $\sum_{\chi \in \Phi^-} \chi(1)^2 = |G| - \sum_{\phi \in \Phi} \phi(1)^2 = 120^2 \cdot 3$. Therefore,

$$|\Phi^-| = 3, \quad k(G) = 15,$$

$$\chi(1) = 120 \quad \text{and} \quad \chi(b) = 1 \quad \text{for all } \chi \in \Phi^-.$$

Thus we have filled two first columns of the character table X of G . By Proposition 1 ((1) $\implies$ (2)), $\phi_{11}(1)\overline{\chi(1)} + |c^G|\phi_{11}(c)\overline{\chi(c)} = 0$ for $\chi \in \Phi^-$. Therefore

$$\chi(c) = -1 \quad \text{for all } \chi \in \Phi^-$$

and hence we know third column of the character table.

At beginning the proof we noted that $\text{rank}(X(\Phi^-, D)) = |\Phi^-|$. Hence, by Proposition 2, $\text{rank}(X(\Phi^-, D^-)) = 0$ (obviously, Φ^- interacts also with D), that is

$$\chi_i(g_j) = 0 \quad \text{for all } i, j.$$

By the second orthogonality relation for the columns corresponding to 1 and d_j ($j \in \{1, 2, 3\}$), we have $1 - 56 \cdot 2 + 77 \cdot 3 + 120 \sum_{i=1}^3 a_{ij} = 0$, where $a_{ij} = \chi_i(g_j)$. Whence the assertion (2) follows.

Let $j \in \{1, 2, 3\}$. Then, by (2), there exists such $i \in \{1, 2, 3\}$ that $\chi_i(d_j) \neq 0$. Since $\chi_i(1) = 120 = 8 \cdot 3 \cdot 5$ is the product of the orders of some Sylow subgroups of G , then as known (see, for example, [2, 5B8]) $\chi_i(g) = 0$ provided $(o(g), 120) \neq 1$. Hence $(o(d_j), 8 \cdot 3 \cdot 5) = 1$. Similarly, from $\phi_6(d_j) \neq 0$ it follows that $(o(d_j), 7 \cdot 11) = 1$. Therefore, $o(d_1) = o(d_2) = o(d_3) = 19$. This and the statement (3) of Lemma 1 imply

$$|C(d_1)| = |C(d_2)| = |C(d_3)| = 19.$$

So, the assertion (1) holds.

Let $j \in \{1, \dots, 9\}$. Suppose that $o(g_j) = 19$. Then $\langle g_j \rangle = \langle d_1 \rangle$ and, as known (for example, by [2, 2A15]), $\chi_i(g_j)^\alpha = a_{ij}$ ($i = 1, 2, 3$) for an automorphism α of the field $\mathbb{Q}(e^{2\pi i/19})$. But it is impossible by (2). Therefore $(o(g_j), 19) = 1$. Similarly, $(o(g_j), 7) = (o(g_j), 11) = 1$. These equations and (1) (see the row “ $|C_G(g)|$ ”) imply (3). $\square$

Lemma 3. *The group G is simple.*

PROOF. Suppose that N is a normal subgroup of G and $\{1\} \neq N \neq G$. Then $N = \bigcap_{\chi \in \Gamma} \text{Ker } \chi$, where $\Gamma \subseteq \text{Irr}(G) \setminus \{\phi_1\}$ and $\text{Ker } \chi = \{g \in G \mid \chi(g) = \chi(1)\}$. Lemma 2 implies that the elements b, c, d_1, d_2, d_3 cannot belong to $\text{Ker } \chi$ for all $\chi \in \text{Irr}(G) \setminus \{\phi_1\}$. Therefore $|N|$ divides $8 \cdot 3 \cdot 5$. Let P be a Sylow subgroup of N . By the Frattini argument, $N_G(P) \supseteq \langle d \rangle$, where $|\langle d \rangle| = 19$. Since $|P| < 19$, $P\langle d \rangle = P \times \langle d \rangle$. But this is a contradiction, since $C_G(d) = \langle d \rangle$ by Lemma 2. $\square$

Lemma 4. (1) *All involutions of G are conjugate.*

(2) *G has an element of order 6.*

(3) *G has at least two conjugacy classes of elements of order 5.*

(4) *G has at least two conjugacy classes of elements of order 10.*

(5) *If G has an element of order 15, the sequence $o(g_1), \dots, o(g_9)$ (in the notation of Lemma 2) within ranking is 2, 3, 5, 5, 6, 10, 10, 15, 15.*

PROOF. (1) Let T be a Sylow 2-subgroup of G and M be a maximal subgroup of T . Since G is simple (by Lemma 3), it follows from [4, IV.1.7] that every involution of T is conjugate with an involution of M . Therefore, assertion (1) will be proved if we shall find a maximal subgroup M in T such that all involutions of M are conjugate in G . We may suppose that T has not cyclic maximal subgroups and hence T is elementary abelian. Then $\text{Aut}(T) \cong GL(3, 2)$ is a group of order $8 \cdot 3 \cdot 7$ without subgroups of order 21. Hence $|N_G(T)/C_G(T)| = n \in \{1, 3, 7\}$. By the Burnside theorem [4, IV.2.6] and simplicity of G , $n \neq 1$. In the case $n = 7$ all involutions of T are conjugate in $N_G(T)$. In the case $n = 3$, T has a maximal subgroup M all involutions of which are conjugate in $N_G(T)$. Thus, (1) holds.

(2) Let g be an element of order 7 of G . Then $\text{Aut}(\langle g \rangle)$ is a cyclic group of order 6 and so $|N_G(\langle g \rangle)/C_G(\langle g \rangle)| = n$ must be a divisor of 6. Since all elements of order 7 in G are conjugate by Lemma 2, all elements of $\langle g \rangle \setminus \{1\}$ are conjugate in $N_G(\langle g \rangle)$ by [4, IV.2.5]. Therefore $n = 6$ and so $N_G(\langle g \rangle)$ has an element s of order 6 ($N_G(\langle g \rangle) = \langle g \rangle \langle s \rangle$ is the Frobenius group of order 42).

(3) Let g be an element of order 5. Then $N_G(\langle g \rangle)/C_G(g)$ is a cyclic group of order $n \in \{2, 4\}$. In the same way as in the proof of (2), we get (considering the element c) that G has an element of order 10. Hence $|C_G(g)|$ is even. If $n = 4$, T has a cyclic quotient group of order 4; therefore, T is abelian and G has a normal 2-complement by [4, IV.2.7]. This contradicts to Lemma 3. Therefore, $n = 2$, whence, by [4, IV.2.5], (3) holds.

(4) This follows easily from (3).

(5) Suppose that G has an element of order 15. It follows from (3) that G has at least two classes of elements of order 15. From this and (1)–(4) it follows that $G \setminus D$ consists of 9 conjugacy classes of G , and the orders of their elements are 2, 3, 5, 5, 6, 10, 10, 15, 15. $\square$

Lemma 5. *Let t be an involution of G . Then $C_G(t) = \langle t \rangle \times F$, where $F \cong A_5$.*

PROOF. Since G has only one conjugacy class of involutions and G has elements of order 6 and 10 (by Lemma 4), $|C_G(t)| = 8 \cdot 3 \cdot 5$ by Lemma 2(3).

Assume that $C_G(t)$ is solvable. Then it has a Hall subgroup of order $3 \cdot 5$ which, clearly, is cyclic. Consequently, G has an element x of order 15 and the element xt of order 30. But this contradicts to the assertion (5) of Lemma 4.

Thus $C_G(t)$ is nonsolvable. Set $C = C_G(t)$. Then $C/\langle t \rangle \cong A_5$ and all involutions of $C/\langle t \rangle$ are conjugate in $C/\langle t \rangle$. Therefore, all (three) subgroups of order 4 of C containing t are conjugate in C . Hence T is elementary abelian or quaternion. If T is elementary abelian then, by [4, IV.2.2], $T \cap Z(C) \cap C' = 1$, whence $t \notin C'$ and $C = \langle t \rangle \times C' \cong \langle t \rangle \times A_5$. The case where T is the quaternion group (and then $C \cong SL(2, 5)$) may be excluded by the result of [3]. $\square$

Now the proof of Theorem follows immediately from Lemmas 3, 4(1), 5 and the theorem of JANKO [5].

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V. A. BELONOgov
INSTITUTE OF MATHEMATICS AND MECHANICS
URAL BRANCH OF RUSSIAN ACADEMY OF SCIENCES
S. KOVALEVSKAYA ST. 16
EKATERINBURG, 620219
RUSSIA

E-mail: belonogov@imm.uran.ru

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