

Operators on quotient indecomposable spaces

By MANUEL GONZÁLEZ (Santander) and JOSÉ M. HERRERA (Santander)

Abstract. We show that a complex Banach space X is a quotient indecomposable Banach space if and only if every operator from X into a quotient space Y of X can be written as $\lambda Q + K$, where $\lambda \in \mathbb{C}$, $Q : X \rightarrow Y$ is the quotient map and K is a strictly cosingular operator. In this way, we obtain a dual version of a result of Ferenczi.

1. Introduction

It has been a long-standing open problem whether every infinite dimensional Banach space X is decomposable; i.e., whether we can write $X = Y \oplus Z$, with Y and Z infinite dimensional closed subspaces. In [6] GOWERS and MAUREY constructed a complex hereditarily indecomposable (HI for short) Banach space, i.e., a space with no decomposable subspaces, which we denote X_{GM} . Moreover, they showed that for a complex HI Banach space X the operators in X have a very simple structure: $L(X) = \mathbb{C}I \oplus SS(X)$ where I is the identity map and $SS(X)$ is the class of all strictly singular operators on X . Later, FERENCZI [1] showed that given a complex HI Banach space X and a subspace $Y \subset X$, we have $L(Y, X) = \mathbb{C}J_Y \oplus SS(Y, X)$, where J_Y is the natural inclusion of Y into X and $SS(Y, X)$ is the class of strictly singular operators from Y into X . Similar results can be obtained in the case of real Banach spaces (see [2, Theorem 2]).

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Infinite dimensional HI spaces may seem an eccentricity, since only recently it has been proved their existence. However, they form an important class of Banach spaces because Gowers has proved the following remarkable result (see [5, Theorem 2]): *Every infinite dimensional Banach space contains an infinite dimensional subspace which either has an unconditional basis or is HI.*

In this paper, we obtain the analogous results for quotient indecomposable (QI for short) Banach spaces, i.e., Banach spaces with no decomposable quotients.

The proof of the above mentioned result for $L(X)$ can be derived from the fact that $L(X)/SS(X)$ is a division algebra when X is HI. So the Gelfand–Mazur theorem applies and the result follows. In our case (as well as in the case of Ferenczi’s result), the main problem is that, in general, there is no product in $L(X, Y)$, neither in $L(X, Y)/SC(X, Y)$, but when X is a QI space we can proceed in a similar way as in Ferenczi’s paper. First of all, we define a relation in the set of all quotients of X , so that given Y, W quotients of X , we say that $W \leq Y$ if the quotient map of W perturbed by a strictly cosingular operator factorises through Y by a surjective operator. Lemma 2 shows that this relation is a partial order and allows us to “project” operators from $L(X, Y)$ to operators in $L(X, W)$, modulo strictly cosingular operators. If we denote $E_Y = L(X, Y)/SC(X, Y)$, then E_Y is a normed space and the previous “projections” define isometries $p_{YW} : E_Y \rightarrow E_W$ for $W \leq Y$, so that $\{E_Y, p_{YW}\}$ is an inductive system and we can consider the inductive limit $\varinjlim E_Y$.

As we observed, there is no natural way of composing operators in $L(X, Y)$, not even in E_Y . However, Lemma 2 solves again the problem. In fact, a natural idea to compose elements $T, T' \in L(X, Y)$ would be to factorise T through Y and then compose with T' . This is not possible in general but passing to further quotients and modulo strictly cosingular operators, such a factorization exists and the desired composition can be defined. Thus, the product can be defined in $\varinjlim E_Y$. Finally, it turns out that $\varinjlim E_Y$ is a normed division algebra and, by the Gelfand–Mazur Theorem, $\varinjlim E_Y$ is isometrically isomorphic to $\mathbb{C}$. Then it is easy to conclude that $E_Y = \mathbb{C}$ for every Y as complex vector spaces and we are done.

We obtain a similar result for the case of real spaces. In particular, we give an algebraic proof of the fact that $L(X)/SC(X)$ is a normed division

algebra. Hence, by the real version of the Gelfand–Mazur Theorem, it is isomorphic to either the reals, or the complexes, or the quaternions (compare with [2, Theorem 7]).

As we mentioned before, the scheme we follow is similar to that of Ferenczi. However, since working with quotients is less intuitive than working with subspaces we have tried to be more explicit in the presentation of the results that we need. Moreover, our scheme has a more algebraic flavour.

Along the paper $X, Y, Z, \dots$ will denote Banach spaces over the field of real or complex numbers. All claims are valid in both fields if there is no explicit mention of one of them. X^* will stand for the dual space of X and $L(X, Y)$ for the linear continuous operators from X into Y ; we set $L(X) = L(X, X)$ and I_X the identity map in X , or simply I if there is no possible confusion. Given $T \in L(X, Y)$ we denote T^* the conjugate operator. Given subspaces $M \subset X$, $N \subset X^*$ we denote, respectively, $M^\perp$, ${}^\perp N$ their annihilators. B_X and $\overset{\circ}{B}_X$ will denote, respectively, the closed and the open unit balls in X . A quotient of a Banach space will always mean a quotient by a closed subspace. We will denote the quotient map of a Banach space X onto a quotient Y by $Q_{XY} : X \rightarrow Y$, or simply by Q_Y if there is no possible confusion.

2. Definitions and basic results

Definition 1. A Banach space X is said to be *decomposable* if it contains a pair of infinite dimensional closed subspaces M and N so that $X = M \oplus N$. Otherwise, X is said to be *indecomposable*.

The space X is said to be *hereditarily indecomposable* (HI for short) if all of its subspaces are indecomposable.

The space X is said to be *quotient indecomposable* (QI for short) if all of its quotients are indecomposable.

It is not difficult to see that if the dual X^* of a Banach space X is HI or QI, then X is QI or HI, respectively. Since the space X_{GM} is HI and reflexive [6], X_{GM}^* is QI.

By the previous remark, our result is straightforward from Ferenczi's result in the case of a reflexive QI space. However, there are examples of non-reflexive QI spaces:

Example. There exists a non reflexive QI Banach space.

Indeed, GOWERS [4] constructed an infinite dimensional Banach space X_G which does not contain c_0 , l_1 , or an infinite dimensional reflexive subspace. Since a non-reflexive subspace of a Banach space with an unconditional basis always contains a copy of c_0 or l_1 [7, Theorem 1.c.13], the Dichotomy Theorem of GOWERS [5, Theorem 2] allows us to conclude that every infinite dimensional subspace of X_G contains an HI subspace Y .

Moreover X_G is a separable dual [4, p. 419]. So, by [7, Proposition 1.b.12], we can assume that Y is contained in a subspace of X_G generated by a boundedly complete basic sequence. Now a perturbation argument allows us to show that Y contains a subspace Z generated by a boundedly complete basic sequence. Hence Z is non reflexive and isomorphic to a dual: $Z \simeq X^*$. Since Z is HI, X is QI.

The following result is well known [11; Proposition 2.2].

Lemma 1. *Let M and N be closed subspaces in a Banach space X . The following assertions are equivalent:*

- a) $X = M + N$.
- b) $M^\perp \cap N^\perp = \{0\}$ and $M^\perp + N^\perp$ is closed.
- c) $\inf\{\|f - g\| : f \in M^\perp, g \in N^\perp, \|f\| = \|g\| = 1\} > 0$.

As a consequence, we derive some characterizations of QI Banach spaces.

Proposition 1. *For a Banach space X the following assertions are equivalent:*

- a) X is quotient indecomposable.
- b) There are no infinite codimensional closed subspaces M and N of X so that we can write $X = M + N$.
- c) Given infinite codimensional closed subspaces M and N of X , for every $\epsilon > 0$ there are $f \in M^\perp$ and $g \in N^\perp$ such that $\|f\| = \|g\| = 1$ and $\|f - g\| < \epsilon$.

PROOF. a) $\implies$ b): $X = M + N$ implies $X/(M \cap N) = M/(M \cap N) \oplus N/(M \cap N)$.

b) $\implies$ a): if $X/M = A \oplus B$ and $Q : X \rightarrow X/M$ is the quotient map, then $X = Q^{-1}(A) + Q^{-1}(B)$.

b) $\iff$ c): follows directly from Lemma 1. $\square$

Recall that an operator $T \in L(X, Y)$ is said to be *strictly cosingular* if there is no quotient map $Q : Y \rightarrow Z$ with Z infinite dimensional, so that QT maps X onto Z . Moreover, T is said to be *lower semi-Fredholm* if its range $R(T)$ is finite codimensional, hence closed [3, Corollary IV.1.13]. We denote by $SC(X, Y)$ and $\Phi_-(X, Y)$ the sets of all strictly cosingular and lower semi-Fredholm operators from X into Y , respectively.

Compact operators are strictly cosingular operators. Moreover, the strictly cosingular and the lower semi-Fredholm operators admit the following characterizations in terms of compact operators.

Theorem 1 (Vladimirski). a) *An operator $T \in L(X, Y)$ is strictly cosingular if and only if for each infinite dimensional quotient map $Q : Y \rightarrow Z$, there is another infinite dimensional quotient map $Q' : Z \rightarrow W$ so that the composition $K = Q'QT : X \rightarrow W$ is compact.*

b) *An operator $T \in L(X, Y)$ is not lower semi-Fredholm if and only if there is an infinite dimensional quotient $Q : Y \rightarrow W$ so that the composition $K = QT : X \rightarrow W$ is compact.*

Moreover, in both cases the quotient W can be chosen in such a way that $\|K\| < \epsilon$ for an arbitrary $\epsilon > 0$.

PROOF. Parts a) and b) are Theorem C.II.6.1 and Lemma C.II.6.3 in [8], respectively. The assertion $\|K\| < \epsilon$ for an arbitrary $\epsilon > 0$ follows directly from the proof of these results. $\square$

Definition 2. Given an operator $T \in L(X, Y)$ we define

$$\begin{aligned}\Gamma_-(T) &= \inf_{Y \rightarrow Z} \|Q_Z T\| \\ \|T\|_q &= \sup_{Y \rightarrow Z} \Gamma_-(Q_Z T)\end{aligned}$$

where the subindex $Y \rightarrow Z$ will mean, from now on, that Z runs over the set of all infinite dimensional quotients of Y .

The following result was proved in [10]. We include a proof for the convenience of the reader.

Proposition 2. a) $\|\cdot\|_q$ is a seminorm on $L(X, Y)$.

b) $\Gamma_-(T) > 0$ if and only if $T \in \Phi_-(X, Y)$.

c) $\|T\|_q = 0$ if and only if $T \in SC(X, Y)$.

PROOF. a) Let $T, U \in L(X, Y)$ and Z a quotient of Y .

From $\|Q_Z(T + U)\| \leq \|Q_Z T\| + \|Q_Z U\|$, taking infima we get

$$\Gamma_-(T + U) \leq \Gamma_-(T) + \|U\|.$$

Applying this to $Q_Z(T + U)$, it follows

$$\Gamma_-(T + U) \leq \Gamma_-(Q_Z(T + U)) \leq \Gamma_-(Q_Z T) + \|Q_Z U\| \leq \|T\|_q + \|Q_Z U\|.$$

Taking infima, we have $\Gamma_-(T + U) \leq \|T\|_q + \Gamma_-(U)$. Now, since $\|Q_Z T\|_q \leq \|T\|_q$, we can write

$$\Gamma_-(Q_Z(T + U)) \leq \|Q_Z T\|_q + \Gamma_-(Q_Z U) \leq \|T\|_q + \Gamma_-(Q_Z U)$$

and taking suprema we conclude $\|T + U\|_q \leq \|T\|_q + \|U\|_q$.

b) If $T \in \Phi_-(X, Y)$, then there exist a finite dimensional subspace N of Y and $\delta > 0$ so that $T(B_X) + N \supset \delta B_Y$. Thus, for every infinite codimensional subspace M of Y , there exists $x \in B_X$ such that $\|x\| = 1$ and $\text{dist}(T(x), M) > \delta/2$; hence $\|Q_{Y/M} T\| > \delta/2$. Conversely, if $T \notin \Phi_-(X, Y)$, then, by Theorem 1, for every $\epsilon > 0$ we can find an infinite dimensional quotient Z of Y such that $\|Q_Z T\| < \epsilon$; hence $\|T\|_q = 0$.

c) If $T \notin SC(X, Y)$, then there is a quotient map $Q_Z : Y \rightarrow Z$ such that $Q_Z T$ is onto. By the open mapping theorem there is $\epsilon > 0$ such that

$$Q_Z T(\overset{\circ}{B}_X) \supset \epsilon \overset{\circ}{B}_Z.$$

Thus, for every quotient map $\phi : Z \rightarrow W$ we have $\|\phi Q_Z T\| \geq \epsilon$. Hence $\|T\|_q \geq \epsilon$ too.

Conversely, if $T \in SC(X, Y)$, it follows from Theorem 1 that $\|T\|_q = 0$. $\square$

Proposition 3. *Let $K \in L(X)$ be a compact operator such that $\|K\| < \epsilon < 1/2$. Then $I + K$ is an isomorphism and $(I + K)^{-1} = I + K_1$, with K_1 compact and $\|K_1\| < 2\epsilon$.*

PROOF. Since $\|K\| < 1/2$, the operator $I + K$ is an isomorphism. Moreover, from $I = (I + K)(I + K)^{-1} = (I + K)^{-1} + K(I + K)^{-1}$, it follows that

$$(1) \quad (I + K)^{-1} = I - K(I + K)^{-1}.$$

Taking norms in (1) we obtain $\|(I + K)^{-1}\| = \|I - K(I + K)^{-1}\| < 1 + \epsilon\|(I + K)^{-1}\|$, hence $\|(I + K)^{-1}\| < \frac{1}{1-\epsilon} < 2$, and if we put $K_1 = -K(I + K)^{-1}$, then we are done. $\square$

3. Operators on quotient indecomposable spaces

In this section X will be a QI Banach space.

We will study the operators from X into any of its quotients using the following relation.

Definition 3. Given two quotients Y, Z of X and the corresponding maps $Q_Y : X \rightarrow Y, Q_Z : X \rightarrow Z$, we write $Z \leq Y$ if there are an operator $K \in SC(X, X)$ and a surjective map $\phi \in L(Y, Z)$ so that $Q_Z(I + K) = \phi Q_Y$.

Our next result shows that the relation $\leq$ defines a filter in the set of all quotients of the space X .

Lemma 2. *Let Y and Z be quotients of X and $0 < \epsilon < 1$. Then there are quotient maps $Y \rightarrow V$ and $Z \rightarrow W$, a compact operator $K : X \rightarrow X$ with $\|K\| < \epsilon$, and an isomorphism $\phi : V \rightarrow W$ so that $Q_W(I - K) = \phi Q_V$. In particular, $W \leq Y, W \leq Z$.*

PROOF. We write $Y = X/M$ and $Z = X/N$. Note that, by Proposition 1, given $\delta > 0$ there are $f \in M^\perp$ and $g \in N^\perp$, so that $\|f\| = \|g\| = 1$ and $\|f - g\| < \delta$. So, taking $\delta = \frac{\epsilon}{2^2}$, we find $f_1 \in M^\perp, g_1 \in N^\perp$, so that $\|f_1\| = \|g_1\| = 1$ and $\|f_1 - g_1\| < \frac{\epsilon}{2^2}$. Then, we choose $x_1 \in X$ such that $\|x_1\| < 2$ and $f_1(x_1) = 1$.

Assume we have selected $f_i \in M^\perp, g_i \in N^\perp$, so that $\|f_i\| = \|g_i\| = 1$ and $x_i \in X$ such that

$$f_i(x_j) = \delta_{ij}; \quad \|x_i\| \|f_i - g_i\| < \frac{\epsilon}{2^i} \quad (i, j \leq n).$$

If we set $F_n = [x_1, \dots, x_n]$ and $G_n = [f_1, \dots, f_n]$, then $\bigcap_{i=1}^n \text{Ker } f_i = {}^\perp G_n$ and we have

$$X = F_n \oplus {}^\perp G_n \quad \text{and} \quad X^* = F_n^\perp \oplus G_n.$$

Let P_n be the projection on X with $\text{Ker } P_n = F_n$ and $R(P_n) = {}^\perp G_n$. Since $M + F_n$ is infinite codimensional and closed, we can choose $f_{n+1} \in (M + F_n)^\perp$ and $g_{n+1} \in N^\perp$ such that

$$\|f_{n+1}\| = \|g_{n+1}\| = 1; \quad \|f_{n+1} - g_{n+1}\| < \frac{\epsilon}{2^{n+2}\|P_n\|}.$$

We take $y_{n+1} \in X$ such that $\|y_{n+1}\| < 2$ and $f_{n+1}(y_{n+1}) = 1$, and set $x_{n+1} = P_n(y_{n+1})$. Since $f_{n+1} \in R(P_n^*) = F_n^\perp$, we have

$$f_{n+1}(x_{n+1}) = (P_n^* f_{n+1})(y_{n+1}) = f_{n+1}(y_{n+1}) = 1.$$

Moreover, $f_{n+1}(x_i) = f_i(x_{n+1}) = 0$ for $i = 1, \dots, n$, by the selection of f_{n+1} and x_{n+1} . And also

$$\|x_{n+1}\| \|f_{n+1} - g_{n+1}\| < \frac{2\|P_n\|\epsilon}{2^{n+2}\|P_n\|} = \frac{\epsilon}{2^{n+1}}.$$

We consider the operator $K : X \rightarrow X$ defined by $K(x) = \sum_{n=1}^\infty (f_n - g_n)(x) x_n$. We have

$$\|K\| \leq \sum_{n=1}^\infty \|f_n - g_n\| \|x_n\| \sum_{n=1}^\infty \frac{\epsilon}{2^n} = \epsilon.$$

Therefore $I - K$ is an isomorphism from X onto X . Moreover, $(I - K)^* f_k = g_k$ for every k . Indeed,

$$\langle K^* f_k, x \rangle = \left\langle f_k, \sum_{n=1}^\infty (f_n - g_n)(x) x_n \right\rangle = \langle f_k - g_k, x \rangle$$

and $K^*(f_k) = f_k - g_k$. From this it follows that

$${}^\perp [g_n] = {}^\perp ((I - K^*)[f_n]) = (I - K)^{-1}({}^\perp [f_n]).$$

Hence $(I - K)({}^\perp [g_n]) = {}^\perp [f_n]$. Now the quotients $V = X/{}^\perp [f_n]$ and $W = X/{}^\perp [g_n]$ fulfil the required conditions. In fact, as $f_i(x_j) = \delta_{ij}$, $[f_n]$ is infinite dimensional so that ${}^\perp [f_n]$ and ${}^\perp [g_n]$ are infinite codimensional. Finally, it is clear that $I - K$ induces an isomorphism ϕ^{-1} from W onto V . $\square$

Lemma 3. *Let Z be a quotient of X . Then for every Banach space Y and every $T \in L(Y, X)$ the quantity $\Gamma_-(Q_Z T)$ does not depend on the choice of Z . Therefore,*

$$\Gamma_-(T) = \|T\|_q \text{ for every } T \in L(Y, X).$$

PROOF. Let Z_1, Z_2 be quotients of X . Given a quotient map $Z_2 \rightarrow W_2$ and $0 < \epsilon < 1$, by Lemma 2 we can find quotient maps $Z_1 \rightarrow W_1, W_2 \rightarrow W'_2$, a compact operator $K : X \rightarrow X$ and an isomorphism $\phi : W_1 \rightarrow W'_2$ such that $\|K\| < \epsilon$ and $\phi Q_{W_1} = Q_{W'_2}(I + K)$.

Moreover, by Proposition 3, we can choose K such that $\|K\| < \epsilon$ and $\|(I + K)^{-1}\| < 1 + 2\epsilon$. Then $\|\phi^{-1}\| = \|\phi^{-1}Q_{W'_2}\| = \|Q_{W_1}(I + K)^{-1}\| \leq \|(I + K)^{-1}\| < 1 + 2\epsilon$, and we have

$$\begin{aligned} \|Q_{W_1}T\| &= \|\phi^{-1}\phi Q_{W_1}T\| \leq \|\phi^{-1}\| \|\phi Q_{W_1}T\| \\ &\leq (1 + 2\epsilon)\|Q_{W'_2}(I + K)T\| \leq (1 + 2\epsilon)(\|Q_{W'_2}T\| + \epsilon\|T\|) \\ &\leq \|Q_{W'_2}T\| + \epsilon(3 + 2\epsilon)\|T\| \leq \|Q_{W_2}T\| + \epsilon' \end{aligned}$$

where $\epsilon' = \epsilon(3 + 2\epsilon)\|T\|$. Thus $\Gamma_-(Q_{Z_1}T) \leq \|Q_{W_2}T\| + \epsilon'$ for any W_2 , hence $\Gamma_-(Q_{Z_1}T) \leq \Gamma_-(Q_{Z_2}T) + \epsilon'$. Since we can take $\epsilon' > 0$ arbitrarily small, $\Gamma_-(Q_{Z_1}T) \leq \Gamma_-(Q_{Z_2}T)$, and by symmetry we are done. $\square$

As an immediate consequence of this lemma and Proposition 2 we obtain the following result.

Corollary 1. *For every Banach space Y we have*

$$L(Y, X) = \Phi_-(Y, X) \cup SC(Y, X).$$

Lemma 4. *Let Y be a quotient of X and let Z be a quotient of Y . If $T \in L(X, Y)$ and $U \in L(Y, Z)$, then*

$$\|UT\|_q \leq \|U\|_q \|T\|_q.$$

PROOF. If $U \in SC(Y, Z)$, then $UT \in SC(X, Z)$; hence $\|UT\|_q = 0 = \|U\|_q \|T\|_q$. Now suppose $U \notin SC(Y, Z)$ or, equivalently by Corollary 1,

that $U \in \Phi_-(Y, Z)$. Then, given $\epsilon > 0$, there exists a quotient $Q_W : Z \rightarrow W$ such that $Q_W U$ is surjective and

$$(1) \quad \|Q_W U\| < \Gamma_-(U) + \epsilon.$$

Let $q : W \rightarrow V$ denote a quotient map with $\dim V = \infty$. Set $N_1 = \text{Ker } Q_W U$ and $N_q = \text{Ker } qQ_W U$, and let $\phi_1 : Y \rightarrow Y/N_1$ and $\phi_q : Y \rightarrow Y/N_q$ be the quotient maps. Then $qQ_W U$ factorises through Y/N_q by an isomorphism $\alpha_q \in L(Y/N_q, V)$ so that the quotients of W correspond in a one to one way with the quotients of Y/N_1 , and the following diagram commutes.

As $qQ_W U = \alpha_q \phi_q$ and $\|qQ_W U\| = \|\alpha_q \phi_q\| = \|\alpha_q\|$, it follows that

$$\|qQ_W UT\| = \|\alpha_q \phi_q T\| \leq \|\alpha_q\| \|\phi_q T\| = \|qQ_W U\| \|\phi_q T\|.$$

Therefore, by Lemma 3, we have

$$\begin{aligned} \Gamma_-(Q_W UT) &= \inf_q \|qQ_W UT\| \leq \inf_q (\|qQ_W U\| \|\phi_q T\|) \\ &\leq \|Q_W U\| \inf_q \|\phi_q T\|. \end{aligned}$$

Now, by Lemma 3, $\inf_q \|\phi_q T\| = \inf_{\psi: Y/N_1 \rightarrow Y/N_q} \|\psi \phi_1 T\| = \Gamma_-(\phi_1 T)$. And by (1), we obtain

$$\Gamma_-(Q_W UT) \leq (\Gamma_-(U) + \epsilon) \Gamma_-(\phi_1 T).$$

Applying Lemma 3 it follows $\|UT\|_q \leq (\|U\|_q + \epsilon)\|T\|_q$ for any $\epsilon > 0$, so we are done. $\square$

For $T \in L(X, Y)$, we denote by $\overline{T}$ the class of T in $L(X, Y)/SC(X, Y)$. Observe that, by Proposition 2, $\|\overline{T}\|_q := \|T\|_q$ defines a norm on $L(X, Y)/SC(X, Y)$. We denote

$$E_Y = (L(X, Y)/SC(X, Y), \|\cdot\|_q).$$

Given $\alpha_Y \in E_Y$ and quotients maps $Q_Y : X \rightarrow Y$, $Q_Z : X \rightarrow Z$ such that $Z \leq Y$, there exist $K \in L(X)$ and a surjective operator $\phi : Y \rightarrow Z$ so that we have $Q_Z(I + K) = \phi Q_Y$. If $T \in L(X, Y)$ is a representative of α_Y , then it is clear that the composition ϕT does not depend on ϕ or T modulo $SC(X, Y)$.

This allows us to define an operator $p_{YZ} : E_Y \rightarrow E_Z$ setting $p_{YZ}(\alpha_Y) = \overline{\phi T}$.

Observe that, if $W \leq Z \leq Y$, then $p_{ZW} p_{YZ} = p_{YW}$.

Lemma 5. *Let Y, Z be quotients of X . Assume that $Z \leq Y$. Then p_{YZ} is a linear isometry.*

PROOF. We proceed in several steps. Let $\alpha_Y \in E_Y$.

First we show that $\|p_{YZ}(\alpha_Y)\|_q \leq \|\alpha_Y\|_q$. Note that we have $p_{YZ}(\alpha_Y) = \overline{\phi T}$. So from Lemma 4 we get

$$\|\phi\|_q = \|\phi Q_Y\|_q = \|Q_Z(I + K)\|_q \leq \|Q_Z\|_q \|I + K\|_q = 1.$$

Therefore $\|\phi\|_q \leq 1$, and again by Lemma 4 we obtain

$$\|p_{YZ}(\alpha_Y)\|_q = \|\phi T\|_q \leq \|\phi\|_q \|T\|_q \leq \|T\|_q = \|\alpha_Y\|_q.$$

In order to prove the remaining inequality, we suppose first that $\phi : Y \rightarrow Z$ is an isomorphism. Then $Z \leq Y$, $Y \leq Z$ and $p_{ZY} p_{YZ} = I_Y$, and we have

$$\|\alpha_Y\|_q = \|p_{ZY} p_{YZ}(\alpha_Y)\|_q \leq \|p_{YZ}(\alpha_Y)\|_q \leq \|\alpha_Y\|_q.$$

So $\|p_{YZ}(\alpha_Y)\|_q = \|\alpha_Y\|_q$.

If ϕ is a quotient map and $\alpha_Y = \overline{T}$, then $p_{YZ}(\alpha_Y) = \overline{\phi T}$. Thus, by Lemma 3,

$$\|p_{YZ}(\alpha_Y)\|_q = \|\phi T\|_q = \|T\|_q = \|\alpha_Y\|_q.$$

The general case, in which ϕ is surjective, follows from the previous cases and the definition of the relation $\leq$. $\square$

Given two quotients Y, Z of X , we will denote by E_{YZ} the set of elements of E_Y that have a representative in $L(X, Y)$ that factorises through Z ; i.e., $\alpha_Y \in E_{YZ}$ if there exist $T \in L(X, Y)$ with $\alpha_Y = \overline{T}$ and $\phi \in L(X, Z)$, $\psi \in L(Z, Y)$ such that $T = \psi\phi$.

Lemma 6. *Let Y, Z be quotients of X and $\alpha_Y \in E_Y$. Then there exists $W \leq Y$ such that $p_{YW}(\alpha_Y) \in E_{WZ}$.*

PROOF. If $\alpha_Y = 0$, then $\alpha_Y \in E_{YZ}$. If $\alpha_Y \neq 0$ and T is a representative of α_Y , then $T \notin SC(X, Y)$, so there exists a quotient $q : Y \rightarrow W$ such that qT is surjective with kernel N . If we consider the natural quotient $Q_{X/N} : X \rightarrow X/N$, by Lemma 1, there are a compact operator $K \in L(X)$ and a quotient $Q : X/N \rightarrow V$ such that $QQ_{X/N}(I + K)$ factorises through Z . Passing to a quotient of W , we can suppose that $V = X/N$ and $Q = I$. Now, if $T' : X/N \rightarrow W$ denotes the map induced by qT , then $T'Q_{X/N} = qT$ and we have

$$\overline{T'Q_{X/N}(I + K)} = \overline{qT} = p_{YW}(\alpha_Y).$$

As $T'Q_{X/N}(I + K)$ factorises through Z , we conclude that $p_{YW}(\alpha_Y) \in E_{WZ}$. $\square$

Since the relation $\leq$ defines a filter in the set of all quotients of X and the maps p_{YZ} are transitive, $\{E_Y, p_{YZ}\}$ is an inductive system and we can consider the inductive limit $\mathbf{E} = \varinjlim E_Y$. We denote by $[\alpha_Y]$ the class in $\varinjlim E_Y$ of an element $\alpha_Y \in E_Y$.

Observe that $\mathbf{E}$ has a natural vector space structure. However, this is not the only structure it has, as we show in the next result.

Theorem 2. *The space $\mathbf{E}$ is a normed division algebra.*

PROOF. We proceed in several steps.

(1) $\mathbf{E}$ is an algebra. Given two elements $\alpha, \beta \in \mathbf{E}$, we choose representatives $\alpha_Y \in E_Y$, $\beta_Z \in E_Z$, of α and β , respectively. By Lemma 6 we can suppose that α_Y has a representative $T \in L(X, Y)$ that factorises through Z by ϕ . If $U \in L(X, Z)$ is a representative of β_Z , then we define the product $\alpha\beta$ as $[\overline{\phi U}] \in \mathbf{E}$.

The definition is correct because it does not depend on the choice of the representatives. The choice of $\beta_Z \in E_Z$ presents no problem. Let α_Y , α_V be representatives of α . By the definition of $\mathbf{E}$, there exists a quotient W of X such that $W \leq Y$, $W \leq V$ and $p_{YW}(\alpha_Y) = p_{VW}(\alpha_V)$. This means that there exist surjective maps $\psi : Y \rightarrow W$, $\psi' : V \rightarrow W$ such that $\overline{\psi T} = \overline{\psi' T'}$ in E_W for representatives T, T' of α_Y, α_V which, by Lemma 6, we can suppose factorise through Z by ϕ, ϕ' , respectively.

The following (partially commutative) diagram may be useful to follow these arguments.

Now, $(\psi\phi - \psi'\phi')Q_Z = \psi T - \psi' T' \in SC(X, W)$ and if we put $h = \psi\phi - \psi'\phi'$, it follows from Lemma 3 that

$$0 = \|hQ_Z\|_q = \Gamma_-(hQ_Z) = \Gamma_-(h) = \|h\|_q.$$

Thus, by Proposition 2, $h = \psi\phi - \psi'\phi' \in SC(Z, W)$. Hence $hU = \psi\phi U - \psi'\phi'U \in SC(X, W)$ and both $\psi\phi U$ and $\psi'\phi'U$ define the same element in E_W , and so in $\mathbf{E}$, as we wanted to prove.

(2) $\mathbf{E}$ is a normed algebra. Given $\alpha \in \mathbf{E}$ and two representatives $\alpha_Y \in E_Y$ and $\beta_Z \in E_Z$ of α , there exists a quotient W of X such that $p_{YW}(\alpha_Y) = p_{ZW}(\beta_Z)$. As p_{YW} and p_{ZW} are isometries (Lemma 5), we have $\|\alpha_Y\|_q = \|\beta_Z\|_q$ and we can define

$$\|\alpha\| := \|\alpha_Y\|_q.$$

All the properties of a norm follow directly from those of $\|\cdot\|_q$ in E_Y . It remains to prove that $\|\alpha\beta\| \leq \|\alpha\| \|\beta\|$.

To see this we take $T \in L(X, Y)$, $U \in L(X, Z)$ such that their equivalence classes in E_Y , E_Z , are representatives of α , β , respectively. Moreover, by Lemma 6, we can suppose, passing to a quotient of Y if necessary, that T factorises through Z by ϕ . By definition, $\alpha\beta = [\overline{\phi U}]$, so it follows

$$\|\alpha\beta\| = \|\phi U\|_q \leq \|\phi\|_q \|U\|_q = \|T\|_q \|U\|_q = \|\alpha_Y\|_q \|\beta_Z\|_q = \|\alpha\| \|\beta\|.$$

(3) $\mathbf{E}$ is a division algebra. Let $\alpha \in \mathbf{E}$, $\alpha \neq 0$ and let α_Y be a representative of α . If $T \in L(X, Y)$ is a representative of α_Y , then $T \notin SC(X, Y)$ and there exists a quotient map $Q : Y \rightarrow Z$ such that $QT : X \rightarrow$

Z is surjective. So, replacing Y by Z and α_Y by $p_{YZ}(\alpha_Y)$ we can suppose that T is surjective and factorises through the quotient $\overline{X} = X/\text{Ker } T$ as $X \rightarrow \overline{X} \xrightarrow{\phi} Z$, i.e. $T = \phi Q_{\overline{X}}$, where ϕ is an isomorphism. Now the inverse of α in $\mathbf{E}$ is $\beta = [\overline{\phi^{-1}Q_Y}]$.

Indeed, by the definition of the product in $\mathbf{E}$ to calculate $\alpha\beta$ we choose the representatives $\overline{T} \in E_Y$, $\overline{\phi^{-1}Q_Y} \in E_{\overline{X}}$ for α , β . As $T = \phi Q_{\overline{X}}$ factorises through $\overline{X}$ ($\overline{T} \in E_{Y\overline{X}}$), we get

$$\alpha\beta = [\overline{\phi\phi^{-1}Q_Y}] = [\overline{Q_Y}] = \mathbf{1},$$

where $\mathbf{1}$ is the unity in $\mathbf{E}$. In the same way, $\phi^{-1}Q_Y$ factorises through Y , and we get

$$\beta\alpha = [\overline{\phi^{-1}T}] = [\overline{Q_{\overline{X}}}] = \mathbf{1}.$$

So the proof is done. $\square$

Proposition 4. *For each quotient Y of X the natural map $p_Y: E_Y \rightarrow \mathbf{E}$, defined as $p_Y(\overline{T}) = [\overline{T}]$, is a linear isometry. Furthermore, when $Y = X$, p_X is an algebra homomorphism; in particular, E_X is a subalgebra of $\mathbf{E}$.*

PROOF. The first part follows from the fact that for any $\alpha_Y \in E_Y$ we have $\|[\alpha_Y]\| = \|\alpha_Y\|_q$. For the second part it is enough to observe that the product in $\mathbf{E}$ of elements represented in E_X is just the composition. $\square$

Now we recall a version of the Gelfand–Mazur Theorem.

Theorem 3 ([9, Theorem 1.7.1]). *Every complex normed division algebra is isomorphic to the field of complex numbers.*

Corollary 2. *If X is a QI complex Banach space, then $\mathbf{E}$ is isometrically isomorphic to the field $\mathbb{C}$ of complex numbers. Moreover, for each Y we have $E_Y = \mathbb{C}$.*

PROOF. By Theorem 2, $\mathbf{E}$ is a normed division algebra with identity $\mathbf{1}$ and $\|\mathbf{1}\| = 1$, so Theorem 3 applies. The second part follows from the first one and Proposition 4. $\square$

We now prove the result claimed in the introduction.

Theorem 4. *Let Z be a complex Banach space. Then Z is QI if and only if for every quotient map $Q_Y : Z \rightarrow Y$ we have*

$$L(Z, Y) = \mathbb{C}Q_Y \oplus SC(Z, Y).$$

PROOF. Suppose that Z is QI . Let Y be a quotient of Z and let $T \in L(Z, Y)$. By Corollary 2, the class of T in $L(Z, Y)/SC(Z, Y)$ is a complex number λ and the class of $T - \lambda Q_Y$ is 0. Therefore, by Proposition 2, $T - \lambda Q_Y \in SC(Z, Y)$.

Conversely, if Z is not QI , then there exists a decomposable quotient Y of Z . Let A and B be two infinite dimensional subspaces of Y such that $Y = A \oplus B$. Let $\pi : Y \rightarrow A$ be the projection of Y onto the first factor and consider the operator $\phi \in L(Z, Y)$ defined by $\phi(x) = \pi(\bar{x})$.

If we assume that $L(Z, Y) = \mathbb{C}Q_Y \oplus SC(Z, Y)$, then $\phi = \lambda Q_Y + K$ for some $0 \neq \lambda \in \mathbb{C}$ and some $K \in SC(Z, Y)$. Now, denoting $D := Q_Y^{-1}(B)$, we get $\phi|_D = 0$. Thus $K|_D = -\lambda(Q_Y)|_D$, which is absurd. $\square$

In the real case, the Gelfand–Mazur Theorem for normed division algebras can be stated as follows.

Theorem 5 ([9, Theorem 1.7.6]). *Every real normed division algebra is isomorphic to either the reals, or the complexes, or the quaternions.*

In this case, we obtain a weaker result from Theorem 2. We denote by $\mathbb{H}$ the quaternions.

Theorem 6. *Let X be a real QI Banach space. Then for every quotient Y of X we have $\dim L(X, Y)/SC(X, Y) \leq 4$.*

Moreover, $L(X)/SC(X)$ is a division algebra isomorphic to either $\mathbb{R}$, or $\mathbb{C}$ or $\mathbb{H}$. In particular, if $L(X)/SC(X)$ is isomorphic to $\mathbb{H}$, then $L(X, Y)/SC(X, Y)$ is also isomorphic to $\mathbb{H}$, for every quotient Y of X .

PROOF. Since by Proposition 4 we can identify the spaces E_Y as subspaces of $\mathbf{E}$, the first part of the theorem follows directly from Theorems 2 and 5.

For the second part, we observe that E_X is a subalgebra of the division algebra $\mathbf{E}$, so it must be a division algebra too (to see this, just consider for each non zero element $a \in E_X$ the linear endomorphisms $h_a(x) = ax$ and $h^a(x) = xa$ on E_X . Since $\mathbf{E}$ has no divisors of zero, h_a and h^a are injective. So they are onto; hence a is invertible). Then, as E_X is a normed division algebra, by the Gelfand–Mazur Theorem (or simply, by the Frobenius Theorem about finite division algebras) it must be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. With the previous identifications we have $E_X \subseteq E_Y \subseteq \mathbb{H}$, so if $E_X = \mathbb{H}$ then for every Y we must have $E_Y = \mathbb{H}$. $\square$

Remark. Note that $E_Y \equiv L(X, Y)/SC(X, Y)$ is not an algebra a priori. However, under the hypothesis of the second part of Theorem 6, this space has the algebra structure translated from $L(X)/SC(X)$. Namely, given any operator $U \in L(X, Y)$ it has the form $Q_Y T_U$, where $T_U \in L(X)$, modulo strictly cosingular operators. Then the product is $\overline{U_1 U_2} = \overline{Q_Y T_{U_1} T_{U_2}}$.

We observe that Corollary 2 also follows from the classical Gelfand–Mazur theorem on complex division Banach algebras, avoiding the use of Theorem 3, which is less natural in the context of Banach algebras.

In this case it remains to check the completeness of $\mathbf{E}$. Next we give a direct proof of this fact, which may have some interest by itself.

Proposition 5. *The normed algebra $\mathbf{E}$ is complete.*

PROOF. It is enough to show that any absolutely converging series in $\mathbf{E}$ is convergent.

In order to do that, we take a sequence $(\alpha_n) \subset \mathbf{E}$ such that $\sum_n \|\alpha_n\| < \infty$, and select representatives $T_n : X \rightarrow X/Y_n$ of α_n such that $\|T_n\| \leq 2\|\alpha_n\|$. Clearly, we can also assume that $Y_0 \subset Y_1 \subset \cdots \subset Y_n \subset \cdots$ and $\dim X/Y_n = \infty$.

First we assume that $Y_\infty := \overline{\bigcup_{n=0}^\infty Y_n}$ is infinite codimensional. We consider the quotient maps $Q_n : X/Y_n \rightarrow X/Y_\infty$ and write $T'_n = Q_n T_n$. Note that $\|T'_n\| \leq \|T_n\| \leq 2\|\alpha_n\|$; hence $\sum_n \|T'_n\| < \infty$. In this way, $\sum_{n=0}^\infty T'_n$ is convergent to $T \in L(X, X/Y_\infty)$ and we have that $\sum_{k=0}^\infty \alpha_k$ converges to $[\overline{T}]$ in $\mathbf{E}$.

It remains to consider the case when Y_∞ is finite codimensional. In this case, adding a few vectors, we can assume that $Y_\infty = X$. We take n_1 such that $Y_{n_1} \not\subset Y_0$, and select $x_1 \in Y_{n_1} \setminus Y_0$ and $f_1 \in Y_0^\perp$ such that $f_1(x_1) = 1$.

Assume that n_i , x_i and f_i have been chosen for $i = 1, \dots, k-1$, and denote $N_k = \bigcap_{i=1}^{k-1} N(f_i)$. We take n_k such that $Y_{n_k} \cap N_k \not\subset Y_{n_{k-1}}$ and select $x_k \in (Y_{n_k} \cap N_k) \setminus Y_{n_{k-1}}$ and $f_k \in Y_{n_{k-1}}^\perp$ such that $f_k(x_k) = 1$. In this way we obtain sequences $(x_k) \subset X$ and $(f_k) \subset X^*$ such that $f_i(x_j) = \delta_{ij}$.

We denote $N = \bigcap_{i=1}^\infty N(f_i)$ and $Q : X \rightarrow X/N$ the quotient map.

Claim. X/N is separable.

Indeed, denoting $Z_k = \bigcap_{i=k+1}^{\infty} N(f_i)$, for every $x \in Z_k$ we have that $x - \sum_{i=1}^k f_i(x)x_i \in N$; i.e., $Z_k = \langle x_1, \dots, x_k \rangle \oplus N$; hence $Q(Z_k)$ is finite dimensional. Moreover, since $Y_{n_k} \subset Z_k$ for every k , we have that $\bigcup_{k=1}^{\infty} Z_k$ is dense in X . Then $\bigcup_{k=1}^{\infty} Q(Z_k)$ is dense in X/N and X/N is separable.

We denote $g_i = \frac{f_i}{\|f_i\|} \in N^{\perp} \equiv (X/N)^*$. The subspace generated by $\{Q(x_i) : i \in \mathbb{N}\}$ is dense in X/N , (g_i) is bounded and $g_i(Q(x_n))$ converges to 0 for any n ; hence g_i converges to 0 with respect to the weak*-topology. Therefore, passing to a subsequence if necessary, a result of Johnson and Rosenthal (see also the Proof of Theorem 1.b.7 in [7]) allows us to assume that (g_i) is a weak*-basic sequence. Thus, by [7; Proposition 1.b.9], $\{Q(x_i) : i \in \mathbb{N}\}$ is a basis in X/N . If C is the corresponding basis constant, then the projections P_n on X/N defined by

$$P_n \left(\sum_{i=1}^{\infty} a_i Q(x_i) \right) = \sum_{i=n+1}^{\infty} a_i Q(x_i)$$

satisfy $\|P_n\| \leq C + 1$, and the operators $\sigma_n : X/Z_n \rightarrow X/N$ defined by $\sigma_n(\bar{x}) = \overline{P_n(x)}$ satisfy $\|\sigma_n\| \leq C + 1$.

Moreover, denoting by $\Pi_n : X/N \rightarrow X/Z_n$ the quotient maps, we have $\Pi_n \sigma_n(x) = x$ for all $x \in X/Z_n$. Now, denoting $T'_n = \sigma_n q_n T_n$, where $q_n : X/Y_n \rightarrow X/Z_n$ is the quotient map, we have that $\|T'_n\| \leq (C+1)\|T_n\|$; thus, $\sum \|T'_n\| < \infty$ and $\sum T'_n$ converges to $T \in L(X, X/N)$. Thus, it is clear that $\sum_{i=0}^n \alpha_i$ converges to $[\bar{T}]$ in $\mathbf{E}$. $\square$

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References

- [1] V. FERENCZI, Operators on subspaces of hereditarily indecomposable Banach spaces, *Bull. London Math. Soc.* **29** (1997), 338–344.
- [2] V. FERENCZI, Hereditarily finitely decomposable Banach spaces, *Studia Math.* **123** (1997), 135–149.
- [3] S. GOLDBERG, Unbounded linear operators, *McGraw-Hill, New York*, 1966.
- [4] W. T. GOWERS, A Banach space not containing c_0 , l_1 or a reflexive subspace, *Trans. Amer. Math. Soc.* **344** (1994), 407–420.
- [5] W. T. GOWERS, A new dichotomy for Banach spaces, *Geometric and Functional Analysis* **6** (1996), 1083–1093.

- [6] W. T. GOWERS and B. MAUREY, The unconditional basic sequence problem, *J. Amer. Math. Soc.* **6** (1993), 851–874.
- [7] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces I–Sequence spaces, *Springer, Berlin, Heidelberg*, 1977.
- [8] D. PRZEWORSKA-ROLEWICZ and S. ROLEWICZ, Equations in Linear Spaces, *PWN-Warszawa*, 1968.
- [9] C. E. RICKART, General Theory of Banach Algebras, *R. E. Krieger. Huntington, New York*, 1960.
- [10] L. WEIS, Über strikt singuläre und strikt cosinguläre Operatoren in Banachräumen, Dokt. Dissert., *Univ. Bonn*, 1974.
- [11] L. WEIS, Perturbation classes of semi-Fredholm operators, *Math. Z.* **178** (1981), 429–442.

MANUEL GONZÁLEZ
DEPARTAMENTO DE MATEMÁTICAS
FACULTAD DE CIENCIAS
UNIVERSIDAD DE CANTABRIA
E-39071 SANTANDER
SPAIN

JOSÉ M. HERRERA
DEPARTAMENTO DE MATEMÁTICAS
FACULTAD DE CIENCIAS
UNIVERSIDAD DE CANTABRIA
E-39071 SANTANDER
SPAIN

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