

A characterization of polynomials through Flett's MVT

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Abstract. We consider the following functional equation

$$f(a) = \sum_{k=0}^n g_k(a - tc)(tc)^k + (\Phi(a) - \Phi(a - c))(tc)^n, \quad a, c \in \mathbb{R}, \quad t \in \mathbb{Q} \cap (0, 1)$$

related to a generalization of the Flett's Mean Value Theorem [4]. We present the solutions of the above equation without any regularity assumptions on the functions $f, g_0, \dots, g_n, \Phi : \mathbb{R} \rightarrow \mathbb{R}$.

1. Introduction

The Lagrange Mean Value Theorem says that for every function $f : [a, b] \rightarrow \mathbb{R}$, continuous on $[a, b]$ and differentiable on (a, b) , there exists a point $c \in (a, b)$ such that

$$f(b) - f(a) = (b - a)f'(c).$$

It is well known that taking $c = \frac{a+b}{2}$ the above equation characterizes quadratic polynomials. J. ACZÉL in [1] and SH. HARUKI in [3] considered a more general equation as follows

$$f(x) - g(y) = h(x + y)(x - y).$$

They proved that f, g, h satisfy this equation if and only if $f(x) = g(x) = ax^2 + bx + c$ and $h(x) = ax + b$, for some $a, b, c \in \mathbb{R}$. Analogously, T. RIEDEL

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and M. SABLIK in [5] solved the functional equation

$$(1) \quad f(c) - f(a) = (c - a)h(c) - \frac{1}{2} \frac{h(b) - h(a)}{b - a} (c - a)^2.$$

Their work was motivated by a generalization of Flett's Mean Value Theorem due to T. RIEDEL and P. K. SAHOO [6].

Theorem. *If $f : [a, b] \rightarrow \mathbb{R}$ is differentiable on $[a, b]$, then there exists a point $c \in (a, b)$ such that*

$$f(c) - f(a) = (c - a)f'(c) - \frac{1}{2} \frac{f'(b) - f'(a)}{b - a} (c - a)^2.$$

T. Riedel and M. Sablik proved that with $c = \frac{a+3b}{4}$ (1) characterizes cubic polynomials.

The following theorem (see [4]) is a generalization of Flett's Mean Value Theorem.

Theorem 1.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be an n times differentiable function. Then there exists a $t = t(a, b) \in (0, 1)$ such that*

$$(2) \quad f(a) = \sum_{k=0}^n \frac{t^k f^{(k)}(a + t(b - a))}{k!} (a - b)^k + \frac{t^{n+1}}{(n+1)!} (f^{(n)}(a) - f^{(n)}(b)) (a - b)^n.$$

If we define functions $g_0, \dots, g_n, \Phi : [a, b] \rightarrow \mathbb{R}$ by

$$(3) \quad g_i(x) = \frac{f^{(i)}(x)}{i!}, \quad i \in \{0, \dots, n\},$$

$$(4) \quad \Phi(x) = \frac{t f^{(n)}(x)}{(n+1)!},$$

then (2) implies that $(f, g_0, \dots, g_n, \Phi)$ satisfies the equation

$$(5) \quad f(a) = \sum_{k=0}^n g_k(a + t(b - a)) (t(a - b))^k + (\Phi(a) - \Phi(b)) (t(a - b))^n.$$

Motivated by [5], [7] we pose two questions about the equation (5).

(i) Find the unknown functions $f, g_0, \dots, g_n, \Phi : \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$(5) \quad f(a) = \sum_{k=0}^n g_k(a + t(b-a))(t(a-b))^k + (\Phi(a) - \Phi(b))(t(a-b))^n.$$

for all $a, b \in \mathbb{R}$ and for a fixed $t \in (0, 1)$.

(ii) Determine if the relations (3), (4) hold for any solution.

The aim of the paper is to answer these questions.

The formula (2) is asymmetric in a, b . One may ask whether Theorem 1.1 is still true if we interchange a and b . The answer is positive.

Theorem 1.1'. *Let $f : [a, b] \rightarrow \mathbb{R}$ be an n times differentiable function. Then there exists an $s = s(a, b) \in (0, 1)$ such that*

$$(2') \quad f(b) = \sum_{k=0}^n \frac{s^k f^{(k)}(b + s(a-b))}{k!} (b-a)^k + \frac{s^{n+1}}{(n+1)!} (f^{(n)}(b) - f^{(n)}(a))(b-a)^n.$$

PROOF. Let us define a transformation $T : [a, b] \rightarrow [a, b]$ by

$$T(x) = a + b - x.$$

Applying Theorem 1.1 to the composition $f \circ T$ completes the proof.

2. Main results

Now we present answers to the forementioned questions. Firstly, we prove a theorem which concerns the connection between polynomials and the set of t 's for which (2) is satisfied.

We begin with some lemmas. The first one easily follows from the binomial formula.

Lemma 2.1. *Let $n \in \mathbb{N}$, $a_i, x, y \in \mathbb{R}$ for $i \in \{0, \dots, n\}$. Then the following formula holds*

$$\sum_{i=0}^n a_i x^i = \sum_{i=0}^n \sum_{k=i}^n \binom{k}{i} a_k (x-y)^{k-i} y^i.$$

We need also the following lemma (cf. [8]).

Lemma 2.2. *Let $N \in \mathbb{N}$ and $C_i : \mathbb{R} \rightarrow \mathbb{R}$, $i \in \{0, \dots, n\}$ be functions homogeneous of i -th order with respect to $k \in \mathbb{N}$, i.e. $C_i(kz) = k^i C_i(z)$. If*

$$\sum_{i=0}^N C_i(z) = 0, \quad \text{then} \quad C_i(z) = 0$$

for every $i \in \{0, \dots, N\}$.

Let us denote by p_N a polynomial of N -th degree, i.e. $p_N = \sum_{i=0}^N d_i x^i$, d_i , $x \in \mathbb{R}$, $i \in \{0, \dots, N\}$, $d_N \neq 0$. We have the following

Theorem 2.3. *Let $n \in \mathbb{N}$ and $f = p_N$ for some $N \in \{0, \dots, n+2\}$. If $N \leq n+1$ then (2) (resp. (2')) holds for every $a, b \in \mathbb{R}$, $a < b$ with any $t \in (0, 1)$ (resp. $s \in (0, 1)$). If $N = n+2$ then (2) (resp. (2')) holds for every $a, b \in \mathbb{R}$, $a < b$, if and only if $t = \frac{n+2}{2n+2}$ (resp. $s = \frac{n+2}{2n+2}$).*

PROOF. Putting $c := a - b$, fix $n \in \mathbb{N}$ and $N \in \{0, \dots, n+2\}$. It is obvious that the k -th derivative of p_N has the following form

$$(6) \quad p_N^{(i)}(x) = \sum_{k=i}^N k(k-1) \cdot \dots \cdot (k-i+1) d_k x^{k-i}.$$

For $f = p_N$ and $N \leq n$ using (6) we get from (2) the following

$$\sum_{i=0}^N d_i a^i = \sum_{i=0}^N \sum_{k=i}^N \binom{k}{i} d_k (a - tc)^{k-i} (tc)^i.$$

The application of Lemma 2.1 ends the proof in this case.

Applying Lemma 2.1 and (6) for $N = n+1$ in (2) we have for every $t \in (0, 1)$

$$\begin{aligned} & \sum_{i=0}^{N-1} \sum_{k=i}^N \binom{k}{i} d_k (a - tc)^{k-i} (tc)^i + \frac{t^N}{N!} (N! d_N a - N! d_N (a - c)) c^{N-1} \\ &= \sum_{i=0}^{N-1} \sum_{k=i}^{N-1} \binom{k}{i} d_k (a - tc)^{k-i} (tc)^i \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=0}^{N-1} \binom{N}{i} d_N(a - tc)^{N-i} (tc)^i + d_N(tc)^N \\
& = \sum_{i=0}^{N-1} d_i a^i + d_N a^N = \sum_{i=0}^N d_i a^i,
\end{aligned}$$

which means that (2) holds with any $t \in (0, 1)$ if $f = p_{n+1}$.

Finally, if $N = n + 2$ the equality (2) yields

$$\begin{aligned}
\sum_{i=0}^N d_i a^i & = \sum_{i=0}^{N-2} \sum_{k=i}^N \binom{k}{i} d_k (a - tc)^{k-i} (tc)^i + \frac{t^{N-1}}{(N-1)!} \\
& \quad \times ((N-1)! d_{N-1} c + 3 \cdot \dots \cdot N d_N (a^2 - (a-c)^2)) c^{N-2} \\
& = \sum_{i=0}^{N-2} \sum_{k=i}^N \binom{k}{i} d_k (a - tc)^{k-i} (tc)^i + d_{N-1} (tc)^{N-1} + N d_N (tc)^{N-1} \left(a - \frac{c}{2}\right) \\
& = \sum_{i=0}^{N-2} \sum_{k=i}^{N-2} \binom{k}{i} d_k (a - tc)^{k-i} (tc)^i + \sum_{i=0}^{N-2} \binom{N-1}{i} d_{N-1} (a - tc)^{N-1-i} (tc)^i \\
& \quad + \sum_{i=0}^{N-2} \binom{N}{i} d_N (a - tc)^{N-i} (tc)^i + d_{N-1} (tc)^{N-1} + N d_N (tc)^{N-1} \left(a - \frac{c}{2}\right).
\end{aligned}$$

Using Lemma 2.1 and the binomial formula we get therefore

$$\begin{aligned}
\sum_{i=0}^N d_i a^i & = \sum_{i=0}^{N-2} d_i a^i + d_{N-1} a^{N-1} \\
& \quad + d_N a^N - N d_N (a - tc) (tc)^{N-1} - d_N (tc)^N + N d_N (tc)^{N-1} \left(a - \frac{c}{2}\right) \\
& = \sum_{i=0}^N d_i a^i - N d_N (a - tc) (tc)^{N-1} - d_N (tc)^N + N d_N (tc)^{N-1} \left(a - \frac{c}{2}\right),
\end{aligned}$$

whence

$$N(a - tc) + tc = N \left(a - \frac{c}{2}\right)$$

and finally

$$t = \frac{N}{2N-2} = \frac{n+2}{2n+2},$$

which ends the proof, since the case of (2') may be treated similarly.

So we have got that polynomials of degree $n + 2$ satisfy (2) with a unique $t = \frac{n+2}{2n+2}$ while for polynomials of lower degree (2) holds for any t from $(0, 1)$. In Theorems 1.1 and 1.1' $t = t(a, b)$ and $s = s(a, b)$ usually are not equal. The above result shows that for polynomials of degree at most $n + 2$, t and s may (or have to, if $N = n + 2$) be equal and do not depend on a and b . Now the question is whether (2) is satisfied only by polynomials if we fix a $t \in (0, 1)$. Observe that in view of the above remarks we admit that (2) holds for any $a, b \in \mathbb{R}$, not necessarily $a < b$.

Putting $c := a - b$ in (5) we obtain the related functional equation in the following form

$$(7) \quad f(a) = \sum_{k=0}^n g_k(a - tc)(tc)^k + (\Phi(a) - \Phi(a - c))(tc)^n, \quad a, c \in \mathbb{R}.$$

Let us introduce a notation. For any $n \in \mathbb{N}$ denote by $SA^n(\mathbb{R}, \mathbb{R})$ the family of all n -additive and symmetric mappings from $\mathbb{R}^n$ into $\mathbb{R}$. If $B \in SA^n(\mathbb{R}, \mathbb{R})$ then B^d will stand for the diagonalization of B . By $B(y^l, z^{l-j})$ we mean a value of B at any l -tuple in which l entries are equal to y and the remaining ones to z .

We are going now to apply Lemma 2.3 from [8] to infer that Φ is a polynomial function. First, observe that any function $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ generates a mapping from $\mathbb{R}$ into $SA^k(\mathbb{R}, \mathbb{R})$, ($k \in \mathbb{N}$) given by

$$\gamma(x)(y_1, \dots, y_k) = \gamma(x)y_1 \cdot \dots \cdot y_k,$$

for every $x, y_1, \dots, y_k \in \mathbb{R}$. Therefore any $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ can be identified with a mapping from $\mathbb{R}$ into $SA^k(\mathbb{R}, \mathbb{R})$ which will be denoted by γ as well. Taking this into account, let us put $x = a$, $y = tc$ and rewrite (7) in the following, equivalent form

$$(7') \quad \Phi(x)(y^n) - f(x) = \sum_{k=0}^n -g_k(x - y)(y^k) + \Phi\left(x - \frac{y}{t}\right)(y^n).$$

Now (7') is a particular case of the equation (4) from Lemma 2.3 in [8]. To see it let us admit (cf. the notation in [8]) $N = n$, $\varphi_N = \Phi$, $\varphi_{N-1} = \dots = \varphi_1 = 0$, $\varphi_0 = -f$, $I_0 = \dots = I_{N-1} = \{(id, -id)\}$, $I_N = \{(id, -id), (id, -\frac{1}{t}id)\}$, $\psi_{k, (id, -id)} = -g_k$, $k \in \{0, \dots, N\}$ and $\psi_{N, (id, -\frac{1}{t}id)} = \Phi$.

Because $t \neq 0$ we have $-\frac{1}{t}id(\mathbb{R}) = \mathbb{R}$, and thus all the assumptions of Lemma 2.3 from [8] are satisfied. In view of the Lemma we see therefore that $\Phi = \varphi_N$ is a polynomial function. It is a well known fact [9] that Φ can be represented in form

$$(8) \quad \Phi = B_0^d + \cdots + B_s^d,$$

where $B_i \in SA^i(\mathbb{R}, \mathbb{R})$, $i \in \{0, \dots, s\}$. Then assuming some regularity on function Φ (e.g. continuity, boundedness on a nonvoid open set, measurability) we conclude that Φ is a polynomial, i.e. Φ has the form $\Phi(x) = b_0 + b_1x + \cdots + b_sx^s$ for some $b_i \in \mathbb{R}$, $i \in \{0, \dots, s\}$. Our aim is to show that this is true even with no regularity assumption. However, we will assume that the fixed t with which (7) holds, is rational.

Let us observe that (7) may be written equivalently with new variables $z := tc$, $y := a - z$ and $u := 1 - \frac{1}{t}$ in the form

$$(9) \quad f(y+z) = \sum_{k=0}^n g_k(y)z^k + (\Phi(y+z) - \Phi(y+uz))z^n.$$

Putting $y := 0$ in (9) we infer

$$(10) \quad f(z) = \sum_{k=0}^n g_k(0)z^k + (\Phi(z) - \Phi(uz))z^n.$$

Taking (8) into account we see that

$$(11) \quad f(z) = \sum_{k=0}^n a_k z^k + (A_1^d(z) + \cdots + A_s^d(z))z^n,$$

where $A_i \in SA^i(\mathbb{R}, \mathbb{R})$, $i \in \{1, \dots, s\}$, while $a_k \in \mathbb{R}$, $k \in \{0, \dots, n\}$ are some constants. Inserting (8) and (11) into (9) and using rational homogeneity of Φ (note that t hence u are rational) we get

$$(12) \quad \begin{aligned} & \sum_{l=0}^n \sum_{i=0}^l \binom{l}{i} a_l y^{l-i} z^i + \sum_{l=1}^s \sum_{j=0}^l \sum_{i=0}^n \binom{l}{j} \binom{n}{i} A_l(y^{l-j}, z^j) y^{n-i} z^i \\ &= \sum_{k=0}^n g_k(y) z^k + \left[\sum_{l=0}^s \sum_{j=0}^l \binom{l}{j} (1-u^j) B_l(y^{l-j}, z^j) \right] z^n. \end{aligned}$$

Suitably arranging terms on both sides of (12) we get the following

$$L_0(y) + \sum_{i=1}^{n+s} L_i(y, z) = R_0(y) + \sum_{i=1}^{n+s} R_i(y, z),$$

where L_i and R_i are those terms on the left and right hand side of (12), respectively, which are homogeneous of degree i in the variable z for every $y, z \in \mathbb{R}$ and $i \in \{0, \dots, n+s\}$. Lemma 2.2 yields $L_i(y, z) = R_i(y, z)$ for $y, z \in \mathbb{R}$ and $i \in \{0, \dots, n+s\}$. In particular we have

$$L_{n+s}(y, z) = A_s(z^s)z^n$$

and

$$R_{n+s}(y, z) = (1 - u^s)B_s(z^s)z^n,$$

whence

$$(13_0) \quad A_s^d(z)z^n = (1 - u^s)B_s^d(z)z^n.$$

It is easy to check that for every $k \in \{1, \dots, s-1\}$

$$L_{n+s-k}(y, z) = \sum_{l=1}^s \sum_{(i,j) \in D_{k,l}} \binom{l}{j} \binom{n}{i} A_l(y^{l-j}, z^j) y^{n-i} z^i,$$

$$R_{n+s-k}(y, z) = \sum_{l=s-k}^s \binom{l}{s-k} B_l(y^{l-s+k}, z^{s-k}) (1 - u^{s-k}) z^n,$$

where $D_{k,l} = \{(i, j) \in \{0, \dots, n\} \times \{0, \dots, l\} : i + j = n + s - k\}$. Obviously the righthand sides of the above formulas are sums of functions $\mathbb{N}$ -homogeneous with respect to y . Applying Lemma 2.2 gives in particular the equality of terms which are homogeneous of k -th order with respect to y , $k \in \{1, \dots, s-1\}$. An easy computation shows that

$$\sum_{(i,j) \in D_{k,s}} \binom{s}{j} \binom{n}{i} A_s(y^{s-j}, z^j) y^{n-i} z^i = \binom{s}{s-k} (1 - u^{s-k}) B_s(y^k, z^{s-k}) z^n.$$

Putting $y := z$ in the above we get

$$\sum_{(i,j) \in D_{k,s}} \binom{s}{j} \binom{n}{i} A_s^d(z) z^n = \binom{s}{s-k} (1 - u^{s-k}) B_s^d(z) z^n.$$

The sum in the above equation may be written equivalently

$$\begin{aligned} \sum_{(i,j) \in D_{k,s}} \binom{s}{j} \binom{n}{i} &= \sum_{i=n-k}^n \binom{s}{n+s-k-i} \binom{n}{i} \\ &= \sum_{i=n-k}^n \binom{s}{k-(n-i)} \binom{n}{n-i} = \sum_{j=0}^k \binom{s}{k-j} \binom{n}{j} = \binom{n+s}{k} \end{aligned}$$

where the last equality follows from the combinatorial meaning of the binomial coefficients. So we have

$$(13_k) \quad A_s^d(z)z^n = \frac{s(s-1) \cdot \dots \cdot (s-k+1)}{(n+s)(n+s-1) \cdot \dots \cdot (n+s-k+1)} \times (1-u^{s-k})B_s^d(z)z^n$$

for $k \in \{1, \dots, s-1\}$. Let us note that the above formula holds also for $k=0$, if we admit the convention that $\prod_{i=p}^q i = 1$ for $q < p$ (cf. (13₀) above). This convention will be used also in the proof of the following lemma.

Lemma 2.4. *Let $f, g_0, \dots, g_n, \Phi : \mathbb{R} \rightarrow \mathbb{R}$ and $t \in \mathbb{Q} \cap (0, 1)$ be such that $(f, g_0, \dots, g_n, \Phi)$ is a solution of (7). Then Φ and f are polynomial functions of degrees at most 2 and $n+2$, respectively.*

PROOF. Suppose that contrary to our claim $s \geq 3$ and consider the equalities (13_{s-3}), (13_{s-2}) and (13_{s-1}). Assuming $B_s \neq 0$ we get

$$\begin{aligned} \frac{s(s-1) \cdot \dots \cdot 2}{(n+s) \cdot \dots \cdot (n+2)}(1-u) &= \frac{s(s-1) \cdot \dots \cdot 3}{(n+s) \cdot \dots \cdot (n+3)}(1-u^2) \\ &= \frac{s(s-1) \cdot \dots \cdot 4}{(n+s) \cdot \dots \cdot (n+4)}(1-u^3) \end{aligned}$$

or

$$(14) \quad \frac{6}{(n+2)(n+3)} = \frac{3}{(n+3)}(1+u) = 1+u+u^2.$$

A simple calculation shows that (14) implies $u^2 < 0$ which contradicts our assumption on s and ends the proof. $\square$

We have proved therefore that (cf. (8) and (10))

$$\Phi(z) = B_0 + B_1(z) + B_2^d(z)$$

and

$$(15) \quad f(z) = \sum_{k=0}^n a_k z^k + (1-u)B_1(z)z^n + (1-u^2)B_2^d(z)z^n$$

for $z \in \mathbb{R}$. In view of Lemma 2.4 and (13₀) we can rewrite (12) in the following way

$$(16) \quad \begin{aligned} & \sum_{l=0}^n \sum_{i=0}^l \binom{l}{i} a_l y^{l-i} z^i + (1-u)(B_1(y) + B_1(z)) \sum_{i=0}^n \binom{n}{i} y^{n-i} z^i \\ & + (1-u^2)B_2^d(y+z) \sum_{i=0}^n \binom{n}{i} y^{n-i} z^i \\ & = \sum_{k=0}^n g_k(y) z^k + (1-u)B_1(z)z^n + 2(1-u)B_2(y, z)z^n + (1-u^2)B_2^d(z)z^n \end{aligned}$$

for $y, z \in \mathbb{R}$. We can arrange terms on both sides of (16) to get sums of $\mathbb{N}$ -homogeneous functions with respect to z . Taking into account summands which are homogeneous of $(n+1)$ -th order in (16) and applying Lemma 2.2 yields

$$\begin{aligned} & (1-u)B_1(z)z^n + 2(1-u^2)B_2(y, z)z^n + (1-u^2)B_2^d(z)nyz^{n-1} \\ & = (1-u)B_1(z)z^n + 2(1-u)B_2(y, z)z^n \end{aligned}$$

for all $y, z \in \mathbb{R}$. Hence we get

$$(17) \quad n(1+u)B_2^d(z)y = -2uB_2(y, z)z$$

for $z \neq 0$. Replacing y by z we get

$$n(1+u)B_2^d(z)z = -2uB_2^d(z)z$$

for $z \neq 0$. Taking $z \in \mathbb{R}$ such that $B_2^d(z) \neq 0$ we have

$$n(1+u) = -2u$$

hence

$$u = -\frac{n}{n+2}$$

or

$$t = \frac{n+2}{2(n+1)}.$$

For $t = \frac{n+2}{2(n+1)}$ we get from (17) for $z \neq 0$

$$B_2^d(z)y = B_2(y,z)z.$$

Dividing by yz^2 and using the symmetry of $B_2(y,z)$ yields

$$\frac{B_2^d(z)}{z^2} = \frac{B_2(y,z)}{yz} = \frac{B_2(z,y)}{zy} = \frac{B_2^d(y)}{y^2}$$

for $y, z \neq 0$. It follows that $\frac{B_2^d(z)}{z^2}$ is a constant, which we denote by b_2 . Thus we get

$$(18) \quad B_2^d(z) = b_2 z^2$$

and this holds for all $z \in \mathbb{R}$ because $B_2^d(0) = 0$. Now we consider terms which are homogeneous of n -th order with respect to z in (16). By Lemma 2.2 and (18) we get

$$\begin{aligned} a_n z^n + (1-u)B_1(y)z^n + n(1-u)B_1(z)y z^{n-1} + (1-u^2)b_2 y^2 z^n \\ + 2n(1-u^2)b_2 y^2 z^n + \binom{n}{n-2}(1-u^2)b_2 y^2 z^n = g_n(y)z^n \end{aligned}$$

for all $z \in \mathbb{R}$, whence for $z \neq 0$

$$\begin{aligned} g_n(y) &= a_n + (1-u)B_1(y) + n(1-u)\frac{B_1(z)y}{z} + (1-u^2)b_2 y^2 \\ &\quad + 2n(1-u^2)b_2 y^2 + \binom{n}{n-2}(1-u^2)b_2 y^2. \end{aligned}$$

Hence for $y, z \neq 0$ we have

$$\begin{aligned} n(1-u)\frac{B_1(z)}{z} &= \frac{1}{y} \left[g_n(y) - a_n - (1-u)B_1(y) - (1-u^2)b_2 y^2 \right. \\ &\quad \left. - 2n(1-u^2)b_2 y^2 - \binom{n}{n-2}(1-u^2)b_2 y^2 \right]. \end{aligned}$$

It is clear that the righthand side does not depend on z . Thus $B_1(z)$ is linear, say

$$(19) \quad B_1(z) := b_1 z$$

for all $z \in \mathbb{R}$. Inserting (18) and (19) in (15) we get

$$(20) \quad f(z) = \begin{cases} \sum_{k=0}^n a_k z^k + \frac{1}{t} b_1 z^{n+1} + \frac{2t-1}{t^2} b_2 z^{n+2}, & \text{if } t = \frac{n+2}{2(n+1)}, \\ \sum_{k=0}^n a_k z^k + \frac{1}{t} b_1 z^{n+1}, & \text{if } t \neq \frac{n+2}{2(n+1)}. \end{cases}$$

Theorem 2.5. *Let $(f, g_0, \dots, g_n, \Phi)$ be a solution of (7), where $t \in \mathbb{Q} \cap (0, 1)$ is fixed. If $t \neq \frac{n+2}{2n+2}$ then Φ and f are polynomials of degrees at most 1 and $n+1$, respectively. If $t = \frac{n+2}{2n+2}$ then the respective degrees are less than or equal to 2 and to $n+2$. Moreover, if $f(x) = \sum_{l=0}^{n+2} a_l x^l$ and $\Phi(x) = b_0 + b_1 x + b_2 x^2$ then*

$$(21) \quad b_1 = t a_{n+1}$$

and

$$(22) \quad b_2 = \begin{cases} 0, & \text{if } t \neq \frac{n+2}{2n+2}, \\ \frac{(n+2)^2}{4(n+1)} a_{n+2}, & \text{if } t = \frac{n+2}{2n+2}. \end{cases}$$

In particular

$$(23) \quad \Phi(x) = c_0 + \frac{t}{(n+1)!} f^{(n)}(x)$$

for some constant c_0 .

PROOF. It remains to prove the last part of the assertion. Defining $a_{n+1} := \frac{1}{t} b_1$ and $a_{n+2} := \frac{2t-1}{t^2} b_2$ and by (20) we get (22). It is now matter of a straightforward calculation to show that (23) holds. $\square$

Now we can present the solution of the equation (7). First however let us prove the following

Lemma 2.6. *Let $f, g_0, \dots, g_n, \Phi : \mathbb{R} \rightarrow \mathbb{R}$ and $t \in \mathbb{Q} \cap (0, 1)$ be such that $(f, g_0, \dots, g_n, \Phi)$ is a solution of (7). Then*

$$g_i(x) = \frac{f^{(i)}(x)}{i!},$$

for $i \in \{0, \dots, n\}$.

PROOF. From the equation (7) and Theorem 2.5 we infer that g is also a polynomial. Indeed, if $f(x) = \sum_{i=0}^{n+2} a_i x^i$ then comparing terms with $(tc)^i$ for $i \in \{0, \dots, n\}$ on both sides of (7) we get

$$\begin{aligned} g_i(y) &= \binom{i}{i} a_i + \binom{i+1}{i} a_{i+1} y + \binom{i+2}{i} a_{i+2} y^2 + \dots \\ &\quad + \binom{n}{i} a_n y^{n-i} + \binom{n+1}{i} a_{n+1} y^{n+1-i} + \binom{n+2}{i} a_{n+2} y^{n+2-i}, \end{aligned}$$

whence we have the following

$$g_0(y) = f(y)$$

and (cf. (6))

$$i!g_i(y) = f^{(i)}(y)$$

which completes the proof. $\square$

Summarizing, our results can be collected in the following

Theorem 2.7. *Let $t \in \mathbb{Q} \cap (0, 1)$ be fixed. The functions $f, g_0, \dots, \dots, g_n, \Phi : \mathbb{R} \rightarrow \mathbb{R}$ satisfy the equation*

$$(7) \quad f(a) = \sum_{k=0}^n g_k(a - tc)(tc)^k + (\Phi(a) - \Phi(a - c))(tc)^n, \quad a, c \in \mathbb{R}$$

if and only if

$$f(x) = \begin{cases} \sum_{i=0}^{n+2} a_i x^i, & \text{if } t = \frac{n+2}{2n+2}, \\ \sum_{i=0}^{n+1} a_i x^i, & \text{if } t \neq \frac{n+2}{2n+2}, \end{cases}, \quad g_i(x) = \frac{f^{(i)}(x)}{i!}$$

for $i \in \{0, \dots, n\}$, and

$$\Phi(x) = c_0 + \frac{t}{(n+1)!} f^{(n)}(x),$$

where a_i , $i \in \{0, \dots, n+2\}$ and c_0 are arbitrary constants.

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