

Some characterizations of π -solvable and supersolvable groups using θ -pairs

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Abstract. For a finite group G , $D_p(G)$ is a generalization of the Frattini subgroup of G . We obtain some results on π -solvable and supersolvable groups with the help of $D_p(G)$ using θ -pairs.

1. Introduction

In the process of developing various conditions characterizing solvable groups, some characteristic groups were defined as the generalization of Frattini subgroup $\phi(G)$ of G . Working in this context in [4] we have introduced a characteristic subgroup $D_p(G)$ and studied its influence on the solvable groups. In [5] N. P. MUKHERJEE and PRABIR BHATTACHARYA obtained some results characterizing supersolvable groups using the class of maximal subgroups M with composite index and $[G : M]_p = 1$ where p is a given prime. In the present paper we obtained a condition characterizing supersolvable groups with the help of θ -pairs introduced by MUKHERJEE and BHATTACHARYA in 1990. Here the maximal subgroups considered are of composite index and the normal index is coprime to p where p is a prime. The family of such maximal subgroups has already been considered in [4]. The paper also contains characterization of π -solvable groups with the help of θ -pairs for a maximal subgroup M in $\delta_p(G)$.

All groups considered here are finite and we have used standard notations as in GORENSTEIN (1968). The notation $M \leq G$ is used to denote that M is a maximal subgroup of G .

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2. Preliminaries

Definition 2.1. Let M be a maximal subgroup of a group G , and H and K two normal subgroups of G with $K \subset H$. The factor group H/K is called a *chief factor* of G if there does not exist any normal subgroup A of G such that $K \subset A \subset H$ with proper inclusion. H is called a *normal supplement* of M in G if $MH = G$. The *normal index* of M in G is defined as the order of a chief factor H/K where H is minimal in the set of normal supplements of M in G , and is denoted by $\eta(G : M)$. It is proved that $\eta(G : M)$ is uniquely determined by M (DESKINS 1959, 2.1) [3].

Definition 2.2. Let G be any group and p any prime. The characteristic subgroup $L(G)$ and $D_p(G)$ are defined as follows:

$$L(G) = \bigcap \{M : M \in \wedge(G)\}, \quad D_p(G) = \bigcap \{M : M \in \delta_p(G)\}$$

where $\wedge(G) = \{M : M \triangleleft G \text{ and } [G : M] \text{ is composite}\}$ and $\delta_p(G) = \{M : M \triangleleft G \text{ and } [G : M] \text{ is composite and } \eta(G : M)_p = 1\}$.

In case $\wedge(G)$ or $\delta_p(G)$ is empty we set $G = L(G)$ or $G = D_p(G)$ respectively.

Theorem 2.3 [2, Theorem 3]. $L(G)$ is supersolvable.

Lemma 2.4 [1, Lemma 2]. If N is a normal subgroup of a group G and M is a maximal subgroup of G such that $N \subseteq M$ then $\eta(G/N : M/N) = \eta(G : M)$.

Definition 2.5 [6]. For a maximal subgroup M of a group G , let $\theta(M) = \{(C, D) : C \leq G, D \triangleleft G, D \subsetneq C, \langle M, C \rangle = G, \langle M, D \rangle = M \text{ and } C/D \text{ contains properly no nontrivial normal subgroup of } G/D\}$.

Lemma 2.6 [6, Lemma 2.1]. If (C, D) is a maximal θ -pair in $\theta(M)$ and $N \triangleleft G, N \subset D$ then $(C/N, D/N)$ is a maximal θ -pair in $\theta(M/N)$ and vice versa.

Definition 2.7. Let L be a non-empty subset of a group G , the *core* of L or *normal interior* of L in G denoted by L_G , is defined to be the join of all the normal subgroups of G that are contained in L , with the convention that $L_G = 1$ if there are no such subgroups. Again $H_G = \bigcap_{g \in G} g^{-1}Hg$ where H is a subgroup.

Lemma 2.8 [3, 2.5]. $[G : M]$ divides $\eta(G : M)$.

Lemma 2.9 [4, Corollary 3.5]. Let N be a normal subgroup of G . If $N \subseteq D_p(G)$ then $D_p(G/N) = D_p(G)/N$.

Theorem 2.10 [4, Theorem 3.6]. If $|D_p(G)|_p = 1$ then G is supersolvable if and only if $G/D_p(G)$ is supersolvable.

Theorem 2.11 [4, Theorem 4.1]. Let p be a prime taken in the definition of $D_p(G)$. Then $D_p(G)$ is solvable if G is a p -solvable group.

Definition 2.12. A finite group G is called p -solvable if it has a subnormal series $1 = V_0 \subset V_1 \subset \cdots \subset V_n = G$ in which each factor group V_{i+1}/V_i , $i = 0, 1, \dots, n-1$, is either a p -group or a p' -group.

Theorem 2.13 [7, Theorem 1]. If M is a maximal subgroup of a group G and M is normal in G then $\eta(G : M) = [G : M] = a$ prime.

3. Some conditions characterizing π -solvable groups

In the present article, we prove some results using θ -pairs in the case when M is a maximal subgroup of composite index such that $\eta(G : M)_p = 1$ where p is a given prime.

Theorem 3.1. Let G be a p -solvable group. G is π -solvable if and only if for each M in $\delta_p(G)$, every maximal θ -pair (C, D) in $\theta(M)$ is such that C/D is π -solvable.

PROOF. Let G be a counter example of minimal order satisfying the hypothesis of the theorem. If $\delta_p(G)$ is empty or G is simple then we can show that G is π -solvable, a contradiction. So $\delta_p(G) \neq \emptyset$ and G is not simple. Let N be a minimal normal subgroup of G . Since G is p -solvable then G/N is p -solvable. We can assume that $\delta_p(G/N)$ is non-empty. Let M/N be any maximal subgroup of G/N in $\delta_p(G/N)$ and $(C/N, D/N)$ be a maximal θ -pair in $\theta(M/N)$. Then by Lemma 2.6, it follows that (C, D) is a maximal θ -pair in $\theta(M)$ where M is in $\delta_p(G)$. Then by the hypothesis C/D is π -solvable. Since $C/N/D/N$ is isomorphic to C/D , it follows that $C/N/D/N$ is π -solvable. Thus G/N satisfies the hypothesis of the theorem. Since $|G/N| < |G|$, G/N is π -solvable. If possible let N_1 be any other minimal normal subgroup of G . Then as above, it can be shown that G/N_1 is π -solvable. Again G , which is isomorphic to a subgroup of

the π -solvable group $G/N \times G/N_1$, is π -solvable, a contradiction. So, we now assume that N is the unique minimal normal subgroup of G . Again as above it can be shown that G/N is π -solvable. Now if $N \subseteq D_p(G)$, then N is solvable and hence N is π -solvable. Consequently, G is π -solvable, a contradiction. We now assume that $N \not\subseteq D_p(G)$. Then there exists M in $\delta_p(G)$ such that $N \not\subseteq M$. So $G = MN$ and $\text{Core}_G M = \langle 1 \rangle$. We claim that $(N, \langle 1 \rangle)$ is a maximal θ -pair in $\theta(M)$. Now $(N, \langle 1 \rangle)$ is a θ -pair in $\theta(M)$ and if possible let (C, D) be a θ -pair such that $(N, \langle 1 \rangle) \subset (C, D)$. Then we must have $D = \langle 1 \rangle$. For, if not, let $D \neq \langle 1 \rangle$. Since M is core free $D \not\subseteq M$. So $G = MD = M$, a contradiction. Then we have $(N, \langle 1 \rangle) \subset (C, \langle 1 \rangle)$ which implies that $N/\langle 1 \rangle = N \subset C = C/\langle 1 \rangle$, again a contradiction as $C/\langle 1 \rangle$ cannot contain any non-trivial normal subgroup of $G/\langle 1 \rangle$. Hence $(N, \langle 1 \rangle)$ is a maximal θ -pair in $\theta(M)$. So by hypothesis $N = N/\langle 1 \rangle$ is π -solvable. Also G/N is π -solvable. Hence G is π -solvable, a contradiction. All these contradictions prove the theorem.

The converse is obvious. $\square$

Theorem 3.2. *Let G be a p -solvable group. G is π -solvable if and only if for each M in $\delta_p(G)$, there exists a normal maximal θ -pair (C, D) in $\theta(M)$ such that C/D is π -solvable.*

PROOF. Let G satisfy the hypothesis of the theorem. If possible let G be a counter example of minimal order. It can be shown that $\delta_p(G)$ is non-empty, and G is not simple. Let N be a minimal normal subgroup of G . Then, we can assume that $\delta_p(G/N) \neq \phi$. Let $M/N \in \delta_p(G/N)$. Then $M \in \delta_p(G)$. By hypothesis there exists a normal maximal θ -pair (C, D) in $\theta(M)$ such that C/D is π -solvable. If $N \subseteq D$ then $(C/N, D/N)$ is a normal maximal θ -pair in $\theta(M/N)$ and $C/N/D/N$ is π -solvable. If $N \not\subseteq D$, we claim that $N \not\subseteq C$. If possible let $N \subseteq C$, then $D \subseteq DN \subseteq C$. Since C/D contains no proper non-trivial normal subgroup of G/D , either $D = DN$ or, $DN = C$. If $D = DN$, then $N \subseteq DN = D$, a contradiction. So $DN = C$. Then $G = \langle M, C \rangle = \langle M, DN \rangle = M$, a contradiction. Hence $N \not\subseteq C$. Now since C/D is π -solvable, CN/DN is also π -solvable. Let K be a maximal proper normal subgroup of G contained in $CN \cap M$ and containing DN . We now claim that CN/K is not a minimal normal subgroup of G/K . For, if not then $(CN, K) \in \theta(M)$. Also $(C, D) \leq (CN, K)$. Since (C, D) is a maximal θ -pair we have $C = CN$. So $N \subseteq C$, a contradiction. Let H/K be a minimal normal subgroup of G/K such

that $H/K \subset CN/K$. We have $H \not\subset M$. So $G = MH$. Therefore (H, K) belongs to $\theta(M)$. Since $DN \subset K \subset H \subset CN$, so H/DN is a subgroup of the π -solvable group CN/DN . Therefore H/DN is π -solvable. Since H/K is an epimorphic image of the π -solvable group H/DN so H/K is π -solvable. Now if (H, K) is a maximal pair in $\theta(M)$ then $(H/N, K/N)$ is a normal maximal pair in $\theta(M/N)$ and $H/N/K/N$ is π -solvable. If (H, K) is not a maximal pair in $\theta(M)$ then let $(H, K) < (H_1, K_1)$ where (H_1, K_1) is a maximal pair in $\theta(M)$ and consequently $H \subset H_1$. Also $K_1 \subsetneq HK_1$. For if $K_1 = HK_1$, then $H \subseteq K_1 \subset M$ and $G = \langle M, H \rangle = M$, a contradiction. Now $HK_1 = H_1$. For if $HK_1 \neq H_1$ then $K_1 \subset HK_1 \subset H_1$ and $HK_1/K_1 \triangleleft G/K_1$ and $HK_1/K_1 \subset H_1/K_1$, a contradiction. Now, $K \subseteq K_1$, so either $K = K_1$ or $K \subset K_1$. If $K = K_1$ then $H_1 = HK_1 = HK = H$, a contradiction. Hence $K \subset K_1$. Again it can be shown that $H/H \cap K_1$ is an epimorphic image of π -solvable group H/K and hence is π -solvable. Since $H_1/K_1 = HK_1/K_1 \cong H/H \cap K_1$ we have H_1/K_1 π -solvable. Thus $(H_1/N, K_1/N)$ is a normal maximal pair in $\theta(M/N)$ by Lemma 2.6, and $H_1/N/K_1/N$ is π -solvable. So by minimality G/N is π -solvable. Now as in Theorem 3.1 we can assume that N is the unique minimal normal subgroup of G . Let $N \subseteq D_p(G)$. Thus N is solvable, so N is π -solvable. So G is π -solvable as G/N is π -solvable. We now suppose that $N \not\subseteq D_p(G)$. Then there exists a core-free maximal subgroup M in $\delta_p(G)$. By hypothesis, there exists a normal maximal θ -pair (C, D) in $\theta(M)$ such that C/D is π -solvable. Since M is Core-free, $D = \langle 1 \rangle$ and consequently C is a minimal normal subgroup of G . By uniqueness of N , we get $N = C$. This implies that N is π -solvable, and hence G is π -solvable, a contradiction. All these contradictions prove the theorem.

The converse is obvious. $\square$

Theorem 3.3. *Let G be a p -solvable group. G is π -solvable if and only if for any two distinct maximal subgroups M_1 and M_2 in $\delta_p(G)$, whenever $\theta(M_1)$ and $\theta(M_2)$ have a common maximal θ -pair (C, D) , it follows that C/D is π -solvable.*

PROOF. Let G be a counter example of minimal order satisfying the hypothesis of the theorem. We can assume that $\delta_p(G) \neq \phi$. Let $\delta_p(G)$ consists of a single element M . Then $D_p(G) = M$. So M is a normal subgroup of G . By Theorem 2.13, we have $\eta(G : M) = [G : M] = a$ prime, a contradiction as $M \in \delta_p(G)$. So we assume that $\delta_p(G)$ consists of at

least two elements, M_1 and M_2 . We can assume that G is not simple. Let N be a minimal normal subgroup of G . As above we can show that $\delta_p(G/N)$ contains more than one element. Let $M_1/N, M_2/N \in \delta_p(G/N)$ and $(C/N, D/N)$ be a common maximal θ -pair in $\theta(M_1/N)$ and $\theta(M_2/N)$. Then M_1 and M_2 are maximal subgroups of G and by Lemma 2.6 we have (C, D) a maximal θ -pair in $\theta(M_1)$ and $\theta(M_2)$. Thus (C, D) is a common maximal θ -pair in $\theta(M_1)$ and $\theta(M_2)$ and by hypothesis we have C/D is π -solvable. Then we have $C/N/D/N$ is π -solvable. Since $|G/N| < |G|$, we have G/N is π -solvable. As in Theorem 3.1 it can be assumed that N is the unique minimal normal subgroup of G . Since G is p -solvable, by Definition 2.12, we have N is either a p -group or a p' -group. If N is a p -group, then it is solvable. This implies that N is π -solvable. Now let N be a p' -group i.e., $|N|_p = 1$. If $N \subseteq D_p(G)$ then N is solvable as $D_p(G)$ is solvable by 2.11. Consequently N is π -solvable. If $N \not\subseteq D_p(G)$, then there exists $M_1 \in \delta_p(G)$ such that $N \not\subseteq M_1$ and so $G = M_1 N$. Then $\eta(G : M_1) = |N|$. Since N is the unique minimal normal subgroup of G , core of M_1 in G is $\langle 1 \rangle$. If $N \subset L(G)$, then N is π -solvable. We now assume that $N \not\subset L(G)$. Then there exists M_2 in $\wedge(G)$ such that $N \not\subseteq M_2$. Then $G = M_2 N$ and $\text{Core}_G M_2 = \langle 1 \rangle$. Also we have $\eta(G : M_2) = |N|$. So $\eta(G : M_2)_p = |N|_p = 1$. As $M_2 \in \wedge(G)$, $[G : M_2]$ is composite. This shows that $M_2 \in \delta_p(G)$. Again as in Theorem 3.1 it can be verified that $(N, \langle 1 \rangle)$ is a common maximal θ -pair in $\theta(M_1)$ and $\theta(M_2)$. By hypothesis $N = N/\langle 1 \rangle$ is π -solvable. Consequently G is π -solvable, a contradiction. All these contradictions prove the theorem. Converse is obvious. $\square$

Theorem 3.4. *Let G be a p -solvable group with a π -solvable maximal subgroup M . Then G is π -solvable if each maximal θ -pair (C, D) in $\theta(M)$ is such that $D_p(G/D) \neq \langle \bar{1} \rangle$, where $\bar{1}$ denotes the identity element of G/D .*

PROOF. Let us consider that G satisfies the hypothesis of the theorem. We assume that G is not simple. Let $H = \text{Core}_G M \neq \langle 1 \rangle$. Since M is a π -solvable maximal subgroup of G , we have M/H is also a π -solvable maximal subgroup of G/H . Let $(C/H, D/H)$ be a maximal θ -pair in $\theta(M/H)$. Then by Lemma 2.6 it follows that (C, D) is a maximal θ -pair in $\theta(M)$. Then by hypothesis we have $D_p(G/D) \neq \langle \bar{1} \rangle$. Since $G/H/D/H \cong G/D$ we have $D_p(G/H/D/H) \neq \langle \bar{1} \rangle$. Thus G/H satisfies the hypothesis of the theorem, so by induction G/H is π -solvable. Again since $H \subseteq M$ and M is π -solvable, H is π -solvable. Hence G is

π -solvable. Let us now suppose that $H = \text{Core}_G M = \langle 1 \rangle$. Let N be a minimal normal subgroup of G . Then $N \not\subseteq M$ and $G = MN$. Since $G/N = MN/N \cong M/M \cap N$ and M is π -solvable, we get G/N is π -solvable. We now assume that N is the unique minimal normal subgroup of G . As in Theorem 3.1, it can be verified that $(N, \langle 1 \rangle)$ is a maximal θ -pair in $\theta(M)$ and so by hypothesis we have $D_p(G) = D_p(G/\langle 1 \rangle) \neq \langle 1 \rangle$. Since $D_p(G)$ is a normal subgroup and N is the unique minimal normal subgroup of G , so, $N \subset D_p(G)$ and hence N is π -solvable. Thus G/N and N are π -solvable which implies that G is π -solvable. Hence the theorem. $\square$

4. A supersolvability condition

In [5] MUKHERJEE and BHATTACHARYA proved some supersolvability conditions for a group where the hypothesis is satisfied by only the maximal subgroups of composite indices. Here we examine a supersolvability condition where the hypothesis is satisfied by maximal subgroups of composite indices of even smaller class.

Theorem 4.1. *Let G be a group with $|D_p(G)|_p = 1$. G is supersolvable if for every maximal subgroup M of $\delta_p(G)$, each θ -pair (C, D) in $\theta(M)$ is such that $D_p(G/D) \neq \langle \bar{1} \rangle$, where $\bar{1}$ denotes the identity element of G/D .*

PROOF. Let $\delta_p(G)$ be empty. Then $G = D_p(G)$. We now claim that $\wedge(G)$ is also empty. If possible, let there exists M in $\wedge(G)$. Then $[G : M]$ is composite. Also by hypothesis $|G|_p = |D_p(G)|_p = 1$. Since $\eta(G : M)$ divides $|G|$, then $\eta(G : M)_p = 1$. Hence M is in $\delta_p(G)$ which contradicts the fact that $\delta_p(G)$ is empty. Hence $\wedge(G)$ is empty and then by definition $G = L(G)$. Since $L(G)$ is supersolvable (by 2.3), G is supersolvable. We now assume that $\delta_p(G)$ has at least one element M . Then G cannot be simple. For, if G is simple, then for each M in $\delta_p(G)$, $(G, \langle 1 \rangle)$ is a θ -pair in $\theta(M)$ and by hypothesis we have $D_p(G/\langle 1 \rangle) \neq \langle \bar{1} \rangle$ i.e., $D_p(G) \neq \langle 1 \rangle$. Since G is simple, we get $G = D_p(G) \subseteq M$, a contradiction. So G is not simple. If $D_p(G) = \langle 1 \rangle$, then for any minimal normal subgroup N of G we have $N \not\subseteq D_p(G)$. This implies that there exists M in $\delta_p(G)$ such that $N \not\subseteq M$, so $G = MN$. It can be shown that $(N, \langle 1 \rangle)$ is a θ -pair in $\theta(M)$ and by hypothesis we have $D_p(G/\langle 1 \rangle) \neq \langle \bar{1} \rangle$ i.e., $D_p(G) \neq \langle 1 \rangle$, a contradiction. So $D_p(G) \neq \langle 1 \rangle$. Let N be a minimal normal subgroup of G such that $N \subseteq D_p(G)$.

Then by Lemma 2.9, $|D_p(G/N)| = |D_p(G)/N| = \frac{|D_p(G)|}{|N|}$. Therefore, $|D_p(G)| = |D_p(G/N)| |N|$. Since $|D_p(G)|_p = 1$, therefore, $|D_p(G/N)|_p = 1$. Let $(C/N, D/N)$ be a θ -pair in $\theta(M/N)$ where M/N is in $\delta_p(G/N)$. Then $M \in \delta_p(G)$. Also by Lemma 2.6 we have (C, D) is a θ -pair in $\theta(M)$. Then by hypothesis we have $D_p(G/D) \neq \langle \bar{1} \rangle$. So $D_p(G/N/D/N) \neq \langle \bar{1} \rangle$. Hence by induction we get G/N is supersolvable. Now an epimorphism $\phi : G/N \rightarrow G/D_p(G)$ can be defined as $\phi(gN) = gD_p(G) \quad \forall g \in G$. So $G/D_p(G)$ is an epimorphic image of the supersolvable group G/N . So $G/D_p(G)$ is supersolvable. Also $|D_p(G)|_p = 1$. So by Theorem 2.10 we have G is supersolvable. Hence the theorem. $\square$

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