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Note on metric spaces and continuous functions

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Abstract. W. Ring, P. Schöpf and J. Schwaiger showed in [RSS] that if E is a finite dimensional normed space then a function $f : E \to \mathbb{R}$ is continuous iff $f \circ \gamma$ is continuous for every regular curve $\gamma : [0, 1] \to E$.

We investigate a similar problem for metric spaces and the class of Lipschitz curves.

1. Introduction

W. Ring, P. Schöpf and J. Schwaiger constructed in [RSS] an example of a not continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f \circ \gamma$ is continuous for every analytic curve $\gamma : (-1, 1) \to \mathbb{R}^2$. They also showed that if instead of analytic we take regular curves, such a function does not exist. In view of the above results the following general problem appears:

Problem 1. Let X, T be metric spaces, let Γ be a family of functions from T into X. We assume that $f : X \to \mathbb{R}$ is such that $f \circ \gamma$ is continuous for every $\gamma \in \Gamma$. Does this imply that f is continuous?

In this paper we investigate the above problem in few cases.

Let us first consider as an illustration the case when X is an arbitrary metric space, T denotes the set $\{0\} \cup \bigcup_{n \in \mathbb{N}} \{\frac{1}{n}\}$, and Γ denotes the space of all continuous functions from T into X.

Let $f : X \to \mathbb{R}$ be arbitrary. We assume that $f \circ \gamma$ is continuous for every $\gamma \in \Gamma$. We show that then f is continuous. For an indirect proof, let us assume that this is not the case. Then there exists an $x_0 \in X$ and a

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sequence $\{x_n\}$ convergent to x_0 such that the sequence $\{f(x_n)\}$ does not converge to $f(x_0)$. We define $\gamma \in \Gamma$ by

$$\gamma(0) = x_0, \quad \gamma\left(\tfrac{1}{n}\right) = x_n.$$

One can now easily notice that $f \circ \gamma$ is not continuous, a contradiction.

Let us now consider a situation when $T = [0,1]$ and Γ is the space of all continuous functions from T into X. Under no additional assumption on X the answer to Problem 1 is negative. It is sufficient to put $X = \{0\} \cup \bigcup_{n\in\mathbb{N}}\{\frac{1}{n}\}$. Then every $\gamma \in \Gamma$ is constant, which means that $f \circ \gamma$ is continuous for every $f : X \to \mathbb{R}$. However, there exist non-continuous functions on X.

As shows the following result, under reasonable assumption on X the answer to Problem 1 is positive.

Theorem 1. *Let X be a locally arcwise connected metric space, and let $T = [0,1]$. Let $f : X \to \mathbb{R}$. If $f \circ \gamma$ is continuous for every continuous function $\gamma : T \to X$ then f is continuous.*

Proof. For an indirect proof let us assume that there exists an $x_0 \in X$ such that f is not continuous at x_0.

Since X is locally arcwise connected for every $n \in \mathbb{N}$ there exists $r_n < \frac{1}{n}$ such that each two points from $B(x_0, r_n)$ can be connected by an arc contained in $B(x_0, \frac{1}{n})$. Without loss of generality we may assume that $\{r_n\}$ is a decreasing sequence.

Since f is not continuous at x_0 there exists a sequence $\{x_n\}$ convergent to x_0 such that $x_n \in B(x_0, r_n)$ and

$$\liminf_{n\to\infty} |f(x_n) - f(x_0)| > 0.$$

Then for every $n \in \mathbb{N}$ there exists a continuous curve $\gamma_n : [0,1] \to B(x_0, \frac{1}{n})$ such that $\gamma_n(0) = x_{n+1}$, $\gamma_n(1) = x_n$. We define a continuous function $\gamma : [0,1] \to X$ by

$$\gamma(t) := \begin{cases} \gamma_n(2^n t - 1) & \text{for } t \in \left[\dfrac{1}{2^n}, \dfrac{1}{2^{n-1}}\right], \ n \in \mathbb{N}, \\ x_0 & \text{for } t = 0. \end{cases}$$

We obtain a contradiction since $f \circ \gamma$ is not continuous at 0. $\square$

Now let us consider as Γ the set of all Lipschitz mappings from $T = [0,1]$ into X. Then the assumption that X is locally arcwise connected does not guarantee a positive solution to Problem 1. As an example one can take as X the graph of an arbitrary continuous nowhere differentiable function $f : [0,1] \to \mathbb{R}$. Then X is locally connected. As there are no non-constant Lipschitz functions $\gamma : [0,1] \to X$, $g \circ \gamma$ is continuous for every function $g : X \to \mathbb{R}$. This suggests that the assumption that X is a locally arcwise connected is too weak, since there there may not exist nontrivial Lipshcitz functions from $[0,1]$ into X. The following definition is an analogue of the definition of locally arcwise connected spaces for Lipschitz curves.

Definition 1. Let X be a metric space. We say that X is *locally Lipschitz connected* if for every point $x \in X$ and $R > 0$ there exists an $r > 0$ such that each points from $B(x,r)$ can be connected by a Lipschitz arc in $B(x,R)$.

It occurs that even this property is two weak to guarantee the positive solution to Problem 1. We have the following result.

Theorem 2. *There exists a compact locally Lipschitz connected metric space $X \subset \mathbb{R}^2$ and a not continuous function $f : X \to \mathbb{R}$ such that $f \circ \gamma$ is continuous for every Lipschitz function $\gamma : [0,1] \to X$.*

Proof. We put $r(x) := |x - \mathrm{round}(x)|$, where $\mathrm{round}(x)$ denotes the nearest integer to x. For $n \in \mathbb{N}$ we define the function $g_n : [0, \frac{1}{2^n}] \to \mathbb{R}^2$ by

$$g_n(x) := \left(\frac{1}{2^n} + \frac{1}{2^n}\sqrt{1 - \frac{1}{4^n}}\, r(4^n x), x \right)$$

and put

$$X_n := g_n\left(\left[0, \frac{1}{2^n}\right]\right), \quad Y := \{(x,0) : x \in [0,1]\}.$$

One can easily check that the g_n is chosen so that the length of the curve g_n is exactly 1. We put $X = \bigcup_{n\geq 0} X_n \cup Y$ (see picture below).

Clearly X is locally Lipschitz connected.

Let $f_n : X_n \to \mathbb{R}$ be defined by

$$f_n(g_n(x)) = 2^n x \quad \text{for } x \in \left[0, \frac{1}{2^n}\right].$$

Figure 1: Set X

We also define $f_0 : Y \to \mathbb{R}$ by $f_0 \equiv 0$. Let $f = \bigcup_{n\geq 0} f_n$. Then $f : X \to \mathbb{R}$ is clearly not continuous at $(0, 0)$.

Let $\gamma : [0, 1] \to X$ be a Lipschitz function. We show that $f \circ \gamma$ is continuous. The function $f \circ \gamma$ is obviously continuous in the neighborhood of every $t \in [0, 1]$ such that $\gamma(t) \neq (0, 0)$. We check what happens in the neighborhood of $(0, 0)$.

Let

$$k_n := \sup \left\{ x \in \left[0, \frac{1}{2^n}\right] : g_n(x) \in \gamma([0, 1]) \right\}.$$

By the definition of g_n the length of the part of γ contained in X_n is greater then $2^n k_n$, which implies that the length of γ is greater then $\sum_n 2^n k_n$. Since length of γ is finite this yields that $2^n k_n$ converges to zero. By (1) this yields that the function f restricted to the set

$$X_\gamma = \bigcup_{n\geq 0} \{g_n(x) : x \in [0, k_n]\} \cup Y$$

is continuous. As $\gamma([0, 1]) \subset X_\gamma$, this implies $f \circ \gamma$ is continuous. $\square$

The reason why such an example can be constructed is that although $(0, 0)$ can be connected with every point x of X by a Lipschitz curve γ_x, the Lipschitz constant of γ_x (as a function of x) is not bounded from above. This leads to the following definition.

Definition 2. Let X be a metric space. We say that X is *uniformly locally Lipschitz connected* if for every point $x \in X$ and $R > 0$ there exist $r > 0$, $L > 0$ such that each points from $B(x, r)$ can be connected by a Lipschitz arc in $B(x, R)$ with Lipschitz constant smaller then L.

We omit the proof of the following result since it is analogous to that of Theorem 1.

Theorem 3. *Let X be a uniformly locally Lipschitz connected metric space, and let $T = [0, 1]$. Let $f : X \to \mathbb{R}$. If $f \circ \gamma$ is continuous for every Lipschitz function $\gamma : T \to X$ then f is continuous.*

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