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Gauss-composition of means and the solution of the Matkowski-Sutô problem

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Abstract. In this paper the so-called Matkowski–Sutô problem is completely solved, that is, continuous and strictly monotonic functions φ and ψ defined on an open real interval I are determined such that the functional equation

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)+\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)=x+y \tag{1}$$

holds for all $x, y \in I$.

The above equation belongs to the class of so-called composite functional equations that does not possess a regularity theory such as known for non-composite equations. The main results of the paper offer new methods to obtain higher-order regularity properties of the unknown functions φ and ψ. First, based on Lebesgue's theorem on the almost everywhere differentiability of monotonic functions, the local Lipschitz property of φ and ψ and their inverses is shown. Then the differentiability of these functions is proved in a subinterval of I. Finally, using Baire's theorem on the continuity properties of derivative functions, the continuous differentiability of φ and ψ in a subinterval is deduced. After these regularity properties, the equation is solved in the subinterval so obtained with earlier methods of the authors. The proof is then completed by using the extension theorem due to the authors and Gy. Maksa.

The main result obtained is a generalization of that of SUTÔ (1914) and MATKOWSKI (1999).

As application, the connection to Gauss composition of means, the equality problem of quasi-arithmetic means and conjugate arithmetic means is discussed and solved.

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Introduction

In 1995 on the 5th International Conference on Functional Equations and Inequalities (Muszyna, Poland) the picnic that is a traditional part of the program was held on a cloudy, rainy day. In the half open log-cabins built for such events participants from all over the world were eating, drinking, talking and later on singing in almost every dense set around the open fireplace. In this surrounding Janusz Matkowski proposed the following *problem* to the Hungarian participants among whom there were a couple of us working in the area of the theory of means: *When will the sum of two quasi-arithmetic means be equal to the double of the arithmetic mean?*

In order to understand the problem, we need to define the notion of *quasi-arithmetic mean* for a non-empty, open interval $I \subset \mathbb{R}$. A function

$M : I^2 \to I$ is called a quasi-arithmetic mean on the interval I if there exists a *strictly monotone* and *continuous* function $\varphi : I \to \mathbb{R}$ such that

$$M(x,y) = \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right) =: A_\varphi(x,y)$$

holds for all $x, y \in I$, where φ^{-1} denotes the (existing) inverse of the function φ. In this case the function φ is called the *generating function* of the quasi-arithmetic mean A_φ. If $\varphi(x) = x$ $(x \in I)$ is the identity function, then the mean generated by φ is the well known *arithmetic mean* defined by

$$A(x,y) = \frac{x+y}{2} \qquad (x, y \in I).$$

Thus, the Matkowski problem is as follows: *Determine all those strictly monotone and continuous functions $\varphi, \psi : I \to \mathbb{R}$ for which the functional equation*

$$\text{(M-S)} \qquad A_\varphi(x,y) + A_\psi(x,y) = x + y \qquad (x, y \in I)$$

holds.

Matkowski noted that he was able to solve the problem only if he made some further (regularity and smoothness) assumptions about the generating functions φ, ψ which are obviously *not natural* since the formulation of the problem does not involve such assumptions.

In that moment many of us inscribed this problem into our minds, and this also meant that we have been thinking about the solution since then with more or less intensity.

The following interesting and important statements about the functional equation (M-S) are fairly clear:

(i) the equation (M-S) is symmetrical, since the roles of the functions φ and ψ, furthermore the variables x and y are interchangeable;

(ii) If the pair φ and ψ is a solution of (M-S) on an interval I, then their restrictions to any nonvoid open subinterval $K \subset I$ are also solutions of (M-S) on interval K;

(iii) The equation (M-S) contains two unknown functions (with given properties) and two independent free variables and it also involves an *iteration* (that is the composition) of the inverses of the unknown

functions, which evidently makes the investigation of this functional equation much harder.

At the same time, the fact that the problem can be stated with relatively few notions is a really nice characteristic, so it can be included among the "elementary" mathematical problems. Based on this last establishment we felt that it was worthwhile to look into the precedents and origins of the problem.

First we came upon the work of O. SUTÔ ([63], [64]) presented in 1914, in which he gave *analytical* solutions for the functional equation (M-S). His result was the same as MATKOWSKI's statement ([40]), who examined the equation by assuming that the functions are twice continuously differentiable.

Searching for preceding works in this topic, we were trying to find out what makes the Matkowski–Sutô problem interesting. On one hand, this is naturally the question of a subjective judgment, on the other hand, we find all those mathematical problems "interesting" that have a considerable history. We have already mentioned the concept of *iteration* (see (iii)). For well-known means, iteration was known from GAUSS' definition of *arithmetic-geometric* mean (*medium arithmeticum-geometricum*) ([20]). He attained this definition by using an iteration on the arithmetic and geometric means. Furthermore, a number of people dealt with the generalization of the Gauss-iteration taking two "abstract" means and examining the convergence of the Gauss-iteration. Based on this result, we formulate the concept of general *mean values* and the definition of Gauss-composition of two means, which, in a special case, includes the Gauss-type arithmetic-geometric mean. The result of the Gauss-composition, if such exists, is again a mean. Since it is true that the Gauss-composition of any two quasi-arithmetic means always exists we can formulate the following problem:

Let us find all those the quasi-arithmetic means whose Gauss-composition is also a quasi-arithmetic mean.

According to the "invariance" equation, this problem turns out to be essentially equivalent to the Matkowski–Sutô problem. This observation opens up further possibilities of research, since an analogous problem can be considered for *classes of mean values* other than the quasi-arithmetic means as well.

The *fifth* of HILBERT's famous problems ([24]) – published in 1900 – can also be mentioned among the preliminaries. Substantially, Hilbert asked if the continuity of certain functions having further algebraic properties, for example, satisfying a functional equation like in the case of Lie's groups, results the differentiability (smoothness) of these functions. This can be looked at as a rather general question and it is still being studied nowadays. JÁNOS ACZÉL ([3]) wrote a summary of this problem, from which it turns out that ANTAL JÁRAI's examinations ([26]–[32]) involving fairly general functional equations *without iteration* may be listed among the solutions of Hilbert's fifth problem. However, when the functional equation does contain iteration, the solution is known only if the equation is of a special type. These can be examined only with very special methods and in these cases the statement, which is implicitly stated in Hilbert's fifth problem, may not hold (see e.g. [6], [7], [39], [38], [53], [54], [56]). In our opinion, the functional equation (M-S) is a possible example of Hilbert's fifth problem, which *contains iteration*, and the result after all proves Hilbert's hypothesis.

Now we will give a brief summary of the history of our research on the Matkowski–Sutô problem. Our first substantial result was achieved at the end of 1998 when we showed that if one of the generating functions of the equation (M-S) is *continuously differentiable*, then the problem can be solved ([18]). Our second important result was the formulation and proof of the extension theorem, which is a joint result with GYULA MAKSA from 1999 ([16]). In this we relied on the solutions of DARÓCZY's special question ([14]) due to C. T. NG ([45]) and M. SABLIK ([55]). Afterwards, our research came to a halt. Although we had some new results with the Matkowski–Sutô problem, these did not give really new methods for answering the original question.

The end of the century was a turning point in our research. At that time we gave up our previous approach and started using the methods that were developed by the second author ([47], [49]) for another area of functional equations. This way we managed to prove further *regularity properties* for the solutions of (M-S). This was possible due to the fact that the equation (M-S) has an implicit monotonicity property. Furthermore, because of the monotonicity property of the unknown functions, we can apply Lebesgue's famous theorem, which states that monotone functions defined on an interval are almost everywhere differentiable. From

this, using nontrivial and very fine methods, we showed that on some non-void subinterval $K \subset I$ *the solutions are differentiable* with nonvanishing derivatives. The last step was to show the *continuity* of the derivatives on some subinterval. The continuity turned out to be the consequence of the Baire Category Theorem, the properties of the functions of Baire Class 1, and a functional equation derived from (M-S) for the derivatives of the unknown functions.

In summary of the historical review we can say that the solution of the Matkowski–Sutô problem is a result of a lot of distressing skepticism (for example in the summer of 2000 Matkowski estimated that the probability of having non smooth solutions was about 50 percent) and some very pleasing recognition and proofs. We recommend that our readers get acquainted with the details of this work by studying our paper.

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1. Gauss-composition of mean values

1.1. Gauss' arithmetic-geometric mean

Let $x, y \in \mathbb{R}_+$ be arbitrary and

$$\begin{aligned} &x_1 := x, && y_1 := y, \\ &x_{n+1} := \frac{x_n + y_n}{2}, && y_{n+1} := \sqrt{x_n y_n} \qquad (n \in \mathbb{N}). \end{aligned} \tag{2}$$

Then there exists the common limit

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n =: A \otimes G(x, y), \tag{3}$$

which is called the arithmetic-geometric mean on the set of the positive real numbers $\mathbb{R}_+$.

The arithmetic-geometric mean played an important role in the history of mathematics. In 1791, the 15-year-old Gauss approximated $A \otimes G(\sqrt{2}, 1)$ to twenty decimal-place accuracy. In 1799, he noticed that the product of the, by that time well known, approximate value of the famous lemniscate constant

$$L := \int_0^1 \frac{dt}{\sqrt{1 - t^4}}$$

and the approximate value of $A\otimes G(\sqrt{2},1)$ coincides with the approximate value of $\frac{\pi}{2}$ to several decimal-place accuracy. On May 30, 1799 he remarked in his diary that if the equation

$$L\cdot A\otimes G(\sqrt{2},1)=\frac{\pi}{2} \tag{4}$$

could be proven, then a new area of mathematics would be born. On December 23, 1799 he wrote in his diary that he had proven the equation (4). Thus, a new route opened to establish the Gauss theory of the elliptic integrals and functions. The events that followed are presented in a number of excellent papers, e.g. in [20], [67], [68], [11], [8], [60], [66].

As it is widely known, Gauss finally found the general formula for $A\otimes G$

$$A\otimes G(x,y)=\left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{dt}{\sqrt{x^2\cos^2 t+y^2\sin^2 t}}\right)^{-1}\qquad (x,y\in\mathbb{R}_+). \tag{5}$$

Since it is desirable for a lot of practical problems to calculate the elliptic integral in (5), which has no exact expression, therefore, even before Gauss, several mathematicians dealt with numerical questions of integrals of this type. Due to the fact that the Gauss-iteration converges quickly to the arithmetic-geometric mean $A\otimes G$, Gauss' discovery plays an important role in numerical analysis as well (cf. [68], [11]).

It has already been realized by Gauss that the mean $A\otimes G$ satisfies the functional equation

$$A\otimes G\left(\frac{x+y}{2},\sqrt{xy}\right)=A\otimes G(x,y) \tag{6}$$

for every $x,y\in\mathbb{R}_+$. The functional equation (6) can be called *invariance equation*. It's crucial role will be shown later on.

The bibliography of the arithmetic-geometric means up to 1927 is given in [20]. Its publisher, H. Geppert, also included his detailed observations on and historical review of Gauss' works.

The generalization of the Gauss-iteration is examined in the following articles [12], [60], [11]. Recently, Matkowski's research has also been pointing in this direction ([40], [41]), but he does not refer to these preliminaries.

After this historical overview, we will give the concept of the Gauss-iteration with the help of the general definition of *means*, and we will examine its convergence. When convergence holds, we define the *Gauss-composition of two means* and consider its invariance property. We will finish this part by looking at some well-known examples.

1.2. Gauss-composition of mean values

Let $I \subset \mathbb{R}$ be a nonvoid open interval.

Definition 1.1. The function $M : I^2 \to I$ is called a *mean* on I if it satisfies the following properties:

(M1) $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$ if $x, y \in I$;

(M2) M is continuous on I^2.

Definition 1.2. The function $M : I^2 \to I$ is called a *strict mean* on I if it is a mean on I and

$$\text{(SM)} \qquad \min\{x, y\} < M(x, y) < \max\{x, y\} \quad \text{if } x \neq y;\ x, y \in I.$$

Let $M_i : I^2 \to I$ $(i = 1, 2)$ be given means on I. Moreover let $(x, y) \in I^2$ be arbitrary. Then the iteration sequence

$$\begin{aligned} &x_1 := x, && y_1 := y, \\ &x_{n+1} := M_1(x_n, y_n), && y_{n+1} := M_2(x_n, y_n) \qquad (n \in \mathbb{N}) \end{aligned} \tag{7}$$

is said to be the *Gauss-iteration* determined by the pair (M_1, M_2) with the initial values $(x, y) \in I^2$.

Let I_n be the closed interval determined by x_n and y_n. Then, because of property (M1) of means, we have

$$I_{n+1} \subseteq I_n \qquad (n \in \mathbb{N}).$$

The Gauss-iteration (7) is said to be *convergent* if the set $\bigcap_{n=1}^{\infty} I_n$ is a singleton for any initial value $(x, y) \in I^2$. By Cantor's theorem, this is true if and only if

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n =: M_1 \otimes M_2(x, y), \tag{8}$$

where $M_1 \otimes M_2 : I^2 \to I$ is a function.

Theorem 1.3. *If M_1 and M_2 are given means on I and the Gauss-iteration determined by the pair (M_1, M_2) is convergent, then the function $M_1 \otimes M_2 : I^2 \to I$ is a mean on I.*

Proof. It is clear that $M_1 \otimes M_2 : I^2 \to I$ satisfies the property (M1), therefore, we only have to prove its continuity.

By their definition the following functions

$$x_n =: M_1^{(n)}(x, y), \qquad y_n = M_2^{(n)}(x, y)$$

are means on I. For $n = 1$, with the notations $x_1 = x =: M_1^{(1)}(x, y)$ and $y_1 = y =: M_2^{(1)}(x, y)$, it is evident. If we suppose that our statement is true for n, then, by (7), we have

$$x_{n+1} = M_1(x_n, y_n) = M_1\big(M_1^{(n)}(x, y), M_2^{(n)}(x, y)\big) =: M_1^{(n+1)}(x, y),$$

$$y_{n+1} = M_2(x_n, y_n) = M_2\big(M_1^{(n)}(x, y), M_2^{(n)}(x, y)\big) =: M_2^{(n+1)}(x, y),$$

whence, by the continuity of the composite function, the statement is true for $(n + 1)$ as well.

Now let

$$\alpha_n(x, y) := \min\{x_n, y_n\}, \quad \omega_n(x, y) := \max\{x_n, y_n\} \qquad (n \in \mathbb{N}).$$

Then $\alpha_n : I^2 \to I$ is a monotone increasing, and $\omega_n : I^2 \to I$ is a monotone decreasing sequence of continuous functions, and both of them converge to the function $M_1 \otimes M_2 : I^2 \to I$. Thus, $M_1 \otimes M_2$ is semi-continuous from below and semi-continuous from above on I^2, therefore, it is continuous. □

Definition 1.4. If M_1 and M_2 are means on I and the Gauss-iteration determined by the pair (M_1, M_2) is convergent, then the uniquely determined mean $M_1 \otimes M_2 : I^2 \to I$ is said to be the *Gauss-composition* of M_1 and M_2 on I.

Theorem 1.5. *If M_1 and M_2 are means on I and one of them is a strict mean on I, then the Gauss-iteration determined by the pair (M_1, M_2) is convergent.*

Proof. We can assume that M_1 is a strict mean on I. Using the notations of the proof of the previous theorem we have that for every

$(x,y)\in I^2$, because of the monotone increasing and monotone decreasing property of $\alpha_n(x,y)$ and $\omega_n(x,y)$ respectively, the following limits exist

$$\alpha(x,y)=\lim_{n\to\infty}\alpha_n(x,y)\quad\text{and}\quad\omega(x,y)=\lim_{n\to\infty}\omega_n(x,y),$$

moreover, $\alpha(x,y)\le\omega(x,y)$.

Contrary to our assumption let us suppose that there exists $(x_0,y_0)\in I^2$ such that $\alpha(x_0,y_0)<\omega(x_0,y_0)$. Then, for every $n\in\mathbb{N}$

$$\alpha_n(x_0,y_0)\le\alpha(x_0,y_0)<\omega(x_0,y_0)\le\omega_n(x_0,y_0). \tag{9}$$

Since M_1 is strict,

$$\alpha(x_0,y_0)<M_1\big(\alpha(x_0,y_0),\omega(x_0,y_0)\big)<\omega(x_0,y_0),$$

$$\alpha(x_0,y_0)<M_1\big(\omega(x_0,y_0),\alpha(x_0,y_0)\big)<\omega(x_0,y_0).$$

Thus, by the continuity of M_1 and by the well-known limit properties, there exists $N\in\mathbb{N}$ such that

$$\alpha(x_0,y_0)<M_1\big(\alpha_N(x_0,y_0),\omega_N(x_0,y_0)\big)<\omega(x_0,y_0),$$

$$\alpha(x_0,y_0)<M_1\big(\omega_N(x_0,y_0),\alpha_N(x_0,y_0)\big)<\omega(x_0,y_0).$$

Because either $x_N=\alpha_N(x_0,y_0)$ or $x_N=\omega_N(x_0,y_0)$, the previous inequalities imply

$$\alpha(x_0,y_0)<x_{N+1}=M_1^{(N+1)}(x_0,y_0)<\omega(x_0,y_0)$$

which contradicts (9). Therefore, our proof is complete. □

Theorem 1.6. *Let M_1 and M_2 be given means on I, and suppose that the Gauss-iteration determined by the pair (M_1,M_2) is convergent. Then, the Gauss-composition $M_1\otimes M_2$ satisfies the invariance equation*

$$M_1\otimes M_2\big(M_1(x,y),M_2(x,y)\big)=M_1\otimes M_2(x,y) \tag{10}$$

for every $x,y\in I$. Furthermore, if $F:I^2\to\mathbb{R}$ is such a continuous function for which $F(x,x)=x$, $(x\in I)$ and it satisfies the functional equation

$$F\big(M_1(x,y),M_2(x,y)\big)=F(x,y) \tag{11}$$

for every $x, y \in I$, then

$$F(x,y) = M_1 \otimes M_2(x,y)$$

for all $x, y \in I$.

Proof. (i) Since $M_1 \otimes M_2$ is a mean on I (Theorem 1.3), therefore,

$$\begin{aligned}\lim_{n\to\infty} M_1 \otimes M_2(x_n, y_n) &= M_1 \otimes M_2\big(M_1 \otimes M_2(x,y), M_1 \otimes M_2(x,y)\big)\\ &= M_1 \otimes M_2(x,y)\end{aligned}$$

for all $x, y \in I$. Thus,

$$M_1 \otimes M_2\big(M_1(x,y), M_2(x,y)\big) = \lim_{n\to\infty} M_1 \otimes M_2(x_{n+1}, y_{n+1}),$$

wherefrom we get (10).

(ii) Assume that $F : I^2 \to \mathbb{R}$ satisfies the stated conditions of the theorem. Then

$$\begin{aligned}\lim_{n\to\infty} F(x_n, y_n) &= F\big(M_1 \otimes M_2(x,y), M_1 \otimes M_2(x,y)\big)\\ &= M_1 \otimes M_2(x,y)\end{aligned}$$

for every $x, y \in I$. But, from (11)

$$\begin{aligned}F(x_{n+1}, y_{n+1}) &= F\big(M_1(x_n, y_n), M_2(x_n, y_n)\big) = F(x_n, y_n)\\ &= F(x_{n-1}, y_{n-1}) = \cdots = F(x,y),\end{aligned}$$

thus,

$$F(x,y) = \lim_{n\to\infty} F(x_n, y_n) = M_1 \otimes M_2(x,y),$$

hence, the proof is complete. □

1.3. Examples for the determination of the Gauss-composition

In this paragraph we are going to recall some known examples from the literature to show applications of our previous theorem.

Example 1.7. Let $I := \mathbb{R}_+$ and

$$M_1(x,y) := \frac{2xy}{x+y}, \qquad M_2(x,y) := \frac{x+y}{2}$$

if $x, y \in \mathbb{R}_+$. Then M_1 is the harmonic and M_2 is the arithmetic mean on $\mathbb{R}_+$ and

$$M_1 \otimes M_2(x,y) = \sqrt{xy} \qquad (x,y \in \mathbb{R}_+). \tag{12}$$

Indeed, the function $F(x,y) := \sqrt{xy}$ $(x,y \in \mathbb{R}_+)$ is continuous and $F(x,x) = x$ (if $x \in \mathbb{R}_+$), furthermore,

$$F\big(M_1(x,y), M_2(x,y)\big) = \sqrt{\frac{2xy}{x+y}\frac{x+y}{2}} = F(x,y),$$

thus, by Theorem 1.6, (12) holds (because M_1 and M_2 are both strict means on I, therefore, the Gauss-iteration is convergent).

Example 1.8. Let $I = \mathbb{R}_+$ and

$$M_1(x,y) := \frac{x+y}{2}, \qquad M_2(x,y) := \sqrt{\frac{x+y}{2}y}$$

for every $x, y \in \mathbb{R}_+$. Then the Gauss-iteration, determined by the pair (M_1, M_2), is convergent, because both means are strict. We have the following statement (Schwab–Borchardt theorem [60], [11]):

$$M_1 \otimes M_2(x,y) = \begin{cases} \dfrac{\sqrt{y^2-x^2}}{\arccos\frac{x}{y}}, & \text{if } 0 < x < y, \\ x, & \text{if } y = x, \\ \dfrac{\sqrt{x^2-y^2}}{\operatorname{area}\cosh\frac{x}{y}}, & \text{if } 0 < y < x. \end{cases}$$

The proof of this formula is equivalent to checking the corresponding invariance equation.

Example 1.9 (Carlson [12]). Let $I = \mathbb{R}_+$ and

$$M_1(x,y) := \sqrt{\frac{x+y}{2}x}, \qquad M_2(x,y) := \sqrt{\frac{x+y}{2}y}$$

for every $x, y \in \mathbb{R}_+$. Then the Gauss-iteration, determined by the pair (M_1, M_2), is convergent, and

$$M_1 \otimes M_2(x,y) = \begin{cases} \sqrt{\dfrac{x^2-y^2}{2\log\left(\frac{x}{y}\right)}}, & \text{if } x \neq y, \\ x, & \text{if } y = x \end{cases}$$

for every $x, y \in \mathbb{R}_+$. The corresponding invariance equation can be checked again by a direct computation.

Further examples for the solution of the invariance equation can be found in [11], [60].

2. Classes of mean values

2.1. Quasi-arithmetic means

The most widely known mean, on a nonvoid open interval $I \subset \mathbb{R}$, is the *arithmetic* mean

$$A(x, y) := \frac{x + y}{2} \qquad (x, y \in I).$$

The generalization of this mean is the well-known concept of the *quasi-arithmetic* mean values. Let $\mathcal{CM}(I)$ denote the class of all continuous and strictly monotone (real valued) functions defined on interval I.

Definition 2.1. The mean $M : I^2 \to I$ is called *quasi-arithmetic* mean on I if there exits $\varphi \in \mathcal{CM}(I)$ such that

$$M(x, y) = \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right) =: A_\varphi(x, y) \tag{13}$$

for every $x, y \in I$-re. In this case $\varphi \in \mathcal{CM}(I)$ is said to be the generating function of the quasi-arithmetic mean (13).

The quasi-arithmetic means constitute the most known class of mean values and there is a rich bibliography relating to them: [1], [5], [2], [4], [23], [19], [25], [34], [43], [52], [65], [33]. Their characterization is solved by the famous Aczél theorem ([1], [2]):

Let $M : I^2 \to I$ be a continuous function having the following properties:

(i) $M(x, x) = x$, *if* $x \in I$,
(ii) $M(x, y) = M(y, x)$, *if* $x, y \in I$,
(iii) $x \mapsto M(x, y)$ *is strictly monotone increasing on I for every fixed* $y \in I$,
(iv) *for all $x, y, u, v \in I$, the bisymmetry equation holds*

$$M\big(M(x, y), M(u, v)\big) = M\big(M(x, u), M(y, v)\big).$$

Then there exists $\varphi \in \mathcal{CM}(I)$ *such that* M *is of the form* (13). *Conversely, if* M *is a quasi-arithmetic mean on* I, *then the properties* (i), (ii), (iii), *and* (iv) *hold and* M *is continuous.*

In the future we will use the notation $M \in \mathcal{QA}(I)$ if $M : I^2 \to I$ is a quasi-arithmetic mean on I.

Definition 2.2. Let $\varphi, \psi \in \mathcal{CM}(I)$. If there exist constants $\alpha \neq 0$ and β such that

$$\psi(x) = \alpha\varphi(x) + \beta \qquad (x \in I), \tag{14}$$

then we say that φ is *equivalent* to ψ on I; in notation: $\varphi(x) \sim \psi(x)$ if $x \in I$ or $\varphi \sim \psi$ on I.

The following result is known ([23], [13]).

Theorem 2.3. *Two quasi-arithmetic means on* I *are equal to each other if and only if their generating functions are equivalent on* I.

That is, if $\varphi, \psi \in \mathcal{CM}(I)$ and $A_\varphi(x, y) = A_\psi(x, y)$ for every $x, y \in I$, then $\varphi(x) \sim \psi(x)$ for $x \in I$, and conversely, if $\varphi \sim \psi$ on I, then $A_\varphi = A_\psi$ on the set I^2.

The most known quasi-arithmetic means are the *homogenous quasi-arithmetic* means. Then $I = \mathbb{R}_+$ and the $M : \mathbb{R}_+^2 \to \mathbb{R}_+$ mean is *homogenous* if

$$M(tx, ty) = t\, M(x, y)$$

holds for every $x, y, t \in \mathbb{R}_+$. It is known, that a quasi-arithmetic mean on $\mathbb{R}_+$ is homogenous if and only if it has the form

$$H_p(x, y) := \begin{cases} \left(\dfrac{x^p + y^p}{2}\right)^{\frac{1}{p}}, & \text{if } p \neq 0 \\ \sqrt{xy} & \text{if } \ p = 0 \end{cases} \qquad (x, y \in \mathbb{R}_+) \tag{15}$$

for some $p \in \mathbb{R}$ (see [23], [48]). (15) is usually called a *power mean* or a *Hölder-mean* in $\mathbb{R}_+$.

Let us note it here, that the notion of *nonsymmetric* quasi-arithmetic mean on I could be defined as it follows. The $M : I^2 \to I$ mean is said to be *nonsymmetric quasi-arithmetic* mean if there exists a constant $0 < \lambda < 1$, $\lambda \neq \frac{1}{2}$ and a function $\varphi \in \mathcal{CM}(I)$ such that

$$M(x, y) = \varphi^{-1}\big(\lambda\varphi(x) + (1 - \lambda)\varphi(y)\big) =: A_\varphi(x, y; \lambda) \quad (x, y \in I). \tag{16}$$

Then λ is called the weight and $\varphi \in \mathcal{CM}(I)$ is said to be the *generating function*. Sometimes (16) is called a *weighted quasi-arithmetic* mean (quasi-arithmetic mean with generating function φ and weight λ).

We note that the (weighted) quasi-arithmetic means are *strict*, so, for any two such means, there *exists* the Gauss-composition. At the same time the arithmetic-geometric mean illustrates that the class of the quasi-arithmetic means is *not closed* for the Gauss-composition, since $A \otimes G$ is not a quasi-arithmetic mean. This statement follows from the corollary of Theorem 2.9 below.

2.2. Some further classes of mean values

Definition 2.4. The $M : I^2 \to I$ mean is called a *conjugate-arithmetic* mean on I if there exists $\varphi \in \mathcal{CM}(I)$ such that

$$M(x,y) = \varphi^{-1}\left(\varphi(x) + \varphi(y) - \varphi\left(\frac{x+y}{2}\right)\right) =: A^*_\varphi(x,y) \tag{17}$$

for every $x, y \in I$. In this case the function $\varphi \in \mathcal{CM}(I)$ is called the generating function of the conjugate-arithmetic mean (17).

Similarly to Theorem 2.3, it is true, that two conjugate-arithmetic means are equal if and only if their generating functions are equivalent ([13]). It is obvious that the conjugate-arithmetic means are strict means. In the case of $I = \mathbb{R}_+$ the homogenous conjugate-arithmetic means are also known (see [13]).

Definition 2.5. The $M : I^2 \to I$ mean is called a *mixed-arithmetic* mean on I if there exists $\varphi \in \mathcal{CM}(I)$ such that

$$M(x,y) = \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y) + \varphi\left(\frac{x+y}{2}\right)}{3}\right) =: A^{\square}_\varphi(x,y) \tag{18}$$

for every $x, y \in I$. As it is usual, $\varphi \in \mathcal{CM}(I)$ is the generating function of (18), which uniquely determines the mixed-arithmetic mean disregarding the equivalence. These means are also strict.

The previously mentioned two classes and the class of the quasi-arithmetic means are contained in the next concept ([17]).

Definition 2.6. Let $\alpha \geq -1$. The mean $M : I^2 \to I$ is said to be a *quasi-arithmetic mean of order* α if there exists $\varphi \in \mathcal{CM}(I)$ such that

$$M(x,y) = \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)+\alpha\varphi\left(\frac{x+y}{2}\right)}{2+\alpha}\right) =: A_{\varphi}^{(\alpha)}(x,y) \tag{19}$$

for every $x, y \in I$. Then the function $\varphi \in \mathcal{CM}(I)$ is called the generating function of (19).

It is clear that for $\alpha = 0$ we have $A_{\varphi}^{(0)} = A_{\varphi}$ (quasi-arithmetic case), for $\alpha = -1$ we have $A_{\varphi}^{(-1)} = A_{\varphi}^{*}$ (conjugate-arithmetic case), and for $\alpha = 1$ we have $A_{\varphi}^{(1)} = A_{\varphi}^{\square}$ (mixed-arithmetic case).

Theorem 2.7. *Let $\alpha \geq -1$. If $\varphi, \psi \in \mathcal{CM}(I)$, then $A_{\varphi}^{(\alpha)}(x,y) = A_{\psi}^{(\alpha)}(x,y)$ for every $(x,y) \in I^2$ if and only if $\varphi \sim \psi$ on I.*

This theorem states that the quasi-arithmetic means of order α are uniquely determined disregarding the equivalence of their generating functions.

Definition 2.8. Let us denote by $\mathcal{P}(I)$ the set of continuous functions $f : I \to \mathbb{R}_+$. The mean $M : I^2 \to I$ is called a *quasi-arithmetic mean weighted with a weight function* if there exist $\varphi \in \mathcal{CM}(I)$ and $f \in \mathcal{P}(I)$ such that

$$M(x,y) = \varphi^{-1}\left(\frac{f(x)\varphi(x)+f(y)\varphi(y)}{f(x)+f(y)}\right) =: A_{\varphi,f}(x,y) \quad (x,y \in I). \tag{20}$$

In the particular case $\varphi(x) = x$ $(x \in I)$, we have the means

$$M(x,y) = \frac{f(x)x+f(y)y}{f(x)+f(y)} =: G_f(x,y) \qquad (x,y \in I)$$

which are called the *Beckenbach–Gini-means* ([9], [10], [21]). These are determined by the weight function $f \in \mathcal{P}(I)$. A characterization theorem for the Beckenbach–Gini means can be found in the paper of Páles and Volkmann ([50]). Now we mention some further special cases:

If $p \in \mathbb{R}$, $I = \mathbb{R}_+$ and $f(x) = x^{p-1}$, then we obtain the mean

$$L_p(x,y) := \frac{x^p+y^p}{x^{p-1}+y^{p-1}} \qquad (x,y \in \mathbb{R}_+)$$

called *Lehmer-means* ([37]).

If $I = \mathbb{R}_+$ and $f(x) := x^r$, $\varphi(x) := x^{s-r}$ $(s \neq r)$, then $A_{\varphi,f}$ reduces to the mean

$$G_{s,r}(x,y) := \begin{cases} \left(\dfrac{x^s + y^s}{x^r + y^r}\right)^{\frac{1}{s-r}}, & \text{if } x,y \in \mathbb{R}_+,\ s \neq r, \\ \exp\left(\dfrac{x^s \ln(x) + y^s \ln(y)}{x^s + y^s}\right), & \text{if } x,y \in \mathbb{R}_+,\ s = r, \end{cases}$$

which is termed the two-parametric *Gini-means* ([21]).

Another class of two parametric means on $I = \mathbb{R}_+$, the so-called *Stolarsky-means* ([61], [62]) are defined as follows:

$$S_{p,q}(x,y) := \begin{cases} \left(\dfrac{q(x^p - y^p)}{p(x^q - y^q)}\right)^{\frac{1}{p-q}}, d & \text{if } pq(p-q)(x-y) \neq 0, \\ \left(\dfrac{x^p - y^p}{p(\ln(x) - \ln(y))}\right)^{\frac{1}{p}}, & \text{if } p(x-y) \neq 0,\ q = 0, \\ \left(\dfrac{q(\ln(x) - \ln(y))}{x^q - y^q}\right)^{-\frac{1}{q}}, & \text{if } q(x-y) \neq 0,\ p = 0, \\ \exp\left(-\dfrac{1}{p} + \dfrac{x^p \ln(x) - y^p \ln(y)}{x^p - y^p}\right), & \text{if } q(x-y) \neq 0,\ p = q, \\ \sqrt{xy}, & \text{if } x - y \neq 0,\ p = q = 0, \\ x, & \text{if } x - y = 0. \end{cases}$$

The mean $S_{0,1}$ is usually called the *logarithmic*, while the mean $S_{1,1}$ is called the *identric* mean. The power mean H_p is also contained in this class since it is easy to check that $S_{2p,p} = H_p$ for all $p \in \mathbb{R}$.

2.3. The generalized Matkowski–Sutô problem

Let $\mathcal{K}$ be a given class of mean values on the open interval $I \subset \mathbb{R}$. We do not define the general concept of the *class of mean values*, instead, we will always think of one of the classes of mean values introduced in Section 2.2. These classes are defined with the help of generating functions, weight functions and parameters. Let us suppose that the elements of $\mathcal{K}$ are strict means on I. Then, for any $M_1, M_2 \in \mathcal{K}$ there exists the Gauss-composition $M_1 \otimes M_2 : I^2 \to I$, which is a mean on I (Theorems 1.3

and 1.5). The Matkowski–Sutô problem for the class of mean values $\mathcal{K}$ is the following: Characterize all those means $M_1, M_2, M_3 \in \mathcal{K}$ for which

$$M_1 = M_2 \otimes M_3 \tag{21}$$

holds, that is, find those means $M_2, M_3 \in \mathcal{K}$ whose Gauss-composition is also a mean belonging to $\mathcal{K}$.

The problem in this form is too general, since it could be asked about any class of mean values $\mathcal{K}$ containing strict means that have not been defined yet. Let us illustrate this on an easy example.

Let $I := \mathbb{R}_+$ and $\mathcal{K} = \{H_p : \mathbb{R}_+^2 \to \mathbb{R}_+ \mid p \in \mathbb{R}\}$, where H_p is the class of power means defined in (15). The elements of this class are strict means on $\mathbb{R}_+$, and depend on a single parameter $p \in \mathbb{R}$. The generalized Matkowski–Sutô problem for power means is to find the parameter triplet $p, q, s \in \mathbb{R}$ such that

$$H_s = H_p \otimes H_q \tag{22}$$

holds on the set $\mathbb{R}_+^2$. In the case $p \neq q$, the solution to this problem was found by Lehmer [37]. For the general case, we have the following result.

Theorem 2.9. *For the triplet $(s, p, q) \in \mathbb{R}^3$ the equation* (22) *holds on the set $\mathbb{R}_+^2$ if and only if*

$$\begin{aligned} &s \neq 0 \quad \textit{then} \quad p = q = s; \quad \textit{and if} \\ &s = 0 \quad \textit{then} \quad p + q = 0. \end{aligned} \tag{23}$$

Proof. (i) First let us suppose that $s \neq 0$. Then, from (22)

$$H_p^s(x, y) + H_q^s(x, y) = x^s + y^s$$

for every $x, y \in \mathbb{R}_+$. Therefore, by the substitutions $u := x^s$, $v := y^s$ and the notations $a := \frac{p}{s}$, $b := \frac{q}{s}$, we have

$$H_a(u, v) + H_b(u, v) = u + v \tag{24}$$

for every $u, v \in \mathbb{R}_+$. Taking the derivative of (24) with respect to u, we have

$$1 = \frac{\partial H_a(u,v)}{\partial u} + \frac{\partial H_b(u,v)}{\partial u} = \frac{1}{2}\left(\left(\frac{u}{H_a(u,v)}\right)^{a-1} + \left(\frac{u}{H_b(u,v)}\right)^{b-1}\right)$$

for every $u, v \in \mathbb{R}_+$. Thus, after the substitution $u = 1$, we get

$$H_a(1,v)^{1-a} + H_b(1,v)^{1-b} = 2. \tag{25}$$

If $a = b$, then, from (22), $a = b = 1$. If $a \neq b$, because of the symmetry, we can assume that $a < b$. Then (24) implies the inequality $a < 1 < b$. Thus,

$$\lim_{v \to +\infty} H_b(1,v)^{1-b} = 0. \tag{26}$$

On the other hand,

$$\lim_{v \to +\infty} H_a(1,v)^{1-a} = \begin{cases} 0 & \text{if } a = 0 \\ +\infty & \text{if } 0 < a < 1 \\ 2^{\frac{a-1}{a}} & \text{if } a < 0, \end{cases}$$

which is a contradiction because of (25) and (26). Therefore, the case $a \neq b$ is not possible, which means $a = b = 1$. Hence, $\frac{p}{s} = \frac{q}{s} = 1$, that is the first part of (23) holds.

(ii) If we assume that $s = 0$, then, by (22),

$$H_p(x,y)H_q(x,y) = xy \tag{27}$$

holds for every $x, y \in \mathbb{R}_+$. Thus, the case $pq = 0$ implies that $p = q = 0$, hence, let us suppose that $pq \neq 0$. Then, by (27), we have

$$H_q(x,y) = H_{-p}(x,y)$$

for every $x, y \in \mathbb{R}_+$, wherefrom $q = -p$ follows. Thus, we have proved the second part of (23) as well. □

Example 1.9 is a special case of this theorem with the choice $s = 0$, $p = 1$, $q = -1$, that is, the Gauss-composition of the arithmetic and harmonic mean is the geometric mean. Let us note that the arithmetic-geometric mean $H_1 \otimes H_0 = A \otimes G$ is evidently not a power mean (moreover, it is not a quasi-arithmetic mean). Because if $A \otimes G$ is a quasi-arithmetic mean on $\mathbb{R}_+$, then, by its homogeneity property, it has to be a power mean. But the triplet $(s, 1, 0) \in \mathbb{R}^3$ does not satisfy any cases of (23), therefore, $A \otimes G$ is not a quasi-arithmetic mean on $\mathbb{R}_+$.

The original Matkowski–Sutô problem is the following. Let M_1 and M_2 be two quasi-arithmetic means on I, and our question is when the Gauss-composition of these means will be equal to the arithmetic mean, that is, when the invariance equation

$$A \circ (M_1, M_2) = A$$

holds. In more details, this means finding those functions $\varphi, \psi \in \mathcal{CM}(I)$ which satisfy the following functional equation

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right) + \psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right) = x + y. \tag{28}$$

To best of our knowledge, the equation (28) was examined first in 1914 by O. Sutô ([63], [64]). He wrote the following about it: "The *analytic* functions φ, ψ and *their inverses* are supposed to be one-valued, as ever we do; $\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)$ is a *mean* of x and y in certain sense." In 1995 the functional equation (28) was rediscovered by Matkowski ([40]), who unambiguously defined the following problem referring to the equation.

Matkowski–Sutô problem: *Find all those functions $\varphi, \psi \in \mathcal{CM}(I)$ for which the functional equation* (28) *holds for every $x, y \in I$.*

The solution of the Matkowski–Sutô problem would give an answer for the following more general Matkowski–Sutô-type problem as well: *Characterize all those quasi-arithmetic means M_1, M_2, and M_3 on I for which the identity*

$$M_1 = M_2 \otimes M_3$$

holds, that is, the Gauss-composition of the quasi-arithmetic means M_2 and M_3 is again a quasi-arithmetic mean on I.

3. The Matkowski–Sutô problem

3.1. The Matkowski–Sutô problem with respect to quasi-arithmetic means

Let $\varphi, \psi \in \mathcal{CM}(I)$. We will say that the pair $(\varphi, \psi) \in \mathcal{CM}(I)^2$ is a *Matkowski–Sutô pair* if the functional equation

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right) + \psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right) = x + y \tag{29}$$

holds for every $x, y \in I$. It is evident that if $(\varphi, \psi) \in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair and $f \sim \varphi$, $g \sim \psi$ on I, then $(f, g) \in \mathcal{CM}(I)^2$ is also a Matkowski–Sutô pair since $A_\varphi = A_f$ and $A_\psi = A_g$ on I^2. Therefore, it is enough to characterize the Matkowski–Sutô pairs disregarding the equivalence of the generating functions.

The first result is due to Sutô ([64]).

Theorem 3.1. *If a pair $(\varphi, \psi) \in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair and the functions φ, ψ are analytic on I, then there exists $p \in \mathbb{R}$ such that*

$$\varphi \sim \chi_p \quad \text{and} \quad \psi \sim \chi_{-p} \qquad \text{on } I \tag{30}$$

where

$$\chi_p(x) := \begin{cases} x & \text{if } p = 0 \\ e^{px} & \text{if } p \neq 0 \end{cases} \qquad (x \in I). \tag{31}$$

The following result is due to Matkowski ([40]). Its regularity assumptions are much weaker than that of Sutô's theorem.

Theorem 3.2. *If a pair $(\varphi, \psi) \in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair and φ, ψ are twice continuously differentiable on I, then there exists $p \in \mathbb{R}$ such that* (30) *holds.*

Our paper [18] improved these two theorems in the following way.

Theorem 3.3. *If a pair $(\varphi, \psi) \in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair, moreover, either φ or ψ is continuously differentiable on I, then there exists $p \in \mathbb{R}$ such that* (30) *holds.*

After this result, we were not able to make further generalization in this direction. To show another possible direction we need the following definition.

Definition 3.4. Let $M, N : I^2 \to I$ be two mean values. We will say that M and N are *strictly comparable* on I if

$$M(x, y) \lhd N(x, y)$$

for every $x, y \in I$, and values $x \neq y$, where $\lhd \in \{ =, <, > \}$ is a relation defined on the set of real numbers.

With the help of the above mentioned concept of comparability Daróczy and Maksa ([15]) proved the following theorem.

Theorem 3.5. *If $(\varphi,\psi)\in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair and the quasi-arithmetic means $M := A_\varphi$ and $N := A_\psi$ are strictly comparable in I, then there exists $p\in\mathbb{R}$ such that* (30) *holds.*

Theorems 3.3 and 3.5 are independent of each other in the following sense: we can not prove directly that if $(\varphi,\psi)\in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair and either φ or ψ is continuously differentiable on I, then A_φ and A_ψ are strictly comparable in I or if $(\varphi,\psi)\in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair and A_φ and A_ψ are strictly comparable in I, then either φ or ψ is continuously differentiable on I.

3.2. Continuously differentiable solutions

Let us suppose that for the $(\varphi,\psi)\in \mathcal{CM}(I)^2$ Matkowski–Sutô pair it is true that φ,ψ are continuously differentiable on I, and $\varphi'(x)\neq 0$ $\psi'(x)\neq 0$ if $x\in I$. This condition is somewhat more special than that of Theorem 3.3, but later we will see that it is enough to solve the Matkowski–Sutô problem.

Theorem 3.6. *If $(\varphi,\psi)\in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair, and φ, ψ are continuously differentiable on I and $\varphi'(x)\neq 0$, $\psi'(x)\neq 0$ for $x\in I$, then there exists $p\in\mathbb{R}$ such that* (30) *holds.*

To prove this theorem we need the following lemmas.

Theorem 3.7. *If $(\varphi,\psi)\in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair and φ, ψ are continuously differentiable on I, and $\varphi'(x)>0$, $\psi'(x)>0$ for $x\in I$, then, with the notations*

$$J := \varphi(I),\quad f := \varphi'\circ\varphi^{-1},\quad g := \psi'\circ\varphi^{-1},$$

the continuous functions $f,g : J\to\mathbb{R}_+$ satisfy the functional equation

$$2f\left(\frac{u+v}{2}\right)\big(g(v)-g(u)\big) = f(u)g(v)-f(v)g(u) \tag{32}$$

for every $u,v\in J$.

Proof. Differentiate the functional equation (29) first with respect to x and then with respect to y. This is possible because of the assumptions

of the lemma, and we have that the equations

$$\frac{\varphi'(x)}{2\varphi'(A_\varphi(x,y))}+\frac{\psi'(x)}{2\psi'(A_\psi(x,y))}=1$$

and

$$\frac{\varphi'(y)}{2\varphi'(A_\varphi(x,y))}+\frac{\psi'(y)}{2\psi'(A_\psi(x,y))}=1$$

hold for every $x,y\in I$. Multiplying the first equation by $\psi'(y)$, the second equation by $\psi'(x)$, and subtracting the new equations from each other, we have

$$\frac{\varphi'(x)\psi'(y)-\varphi'(y)\psi'(x)}{2\varphi'(A_\varphi(x,y))}=\psi'(y)-\psi'(x) \tag{33}$$

for every $x,y\in I$. Let $u=\varphi(x)$, $v=\varphi(y)$ $(u,v\in J:=\varphi(I))$ be arbitrary and $f:=\varphi'\circ\varphi^{-1}$, $g:=\psi'\circ\varphi^{-1}$, then from (33) we get that

$$2f\left(\frac{u+v}{2}\right)\big(g(v)-g(u)\big)=f(u)g(v)-f(v)g(u)$$

holds for every $u,v\in J$, that is, the functional equation (32) is satisfied. In this case, because of the assumptions stated in the lemma, the functions $f,g:J\to\mathbb{R}_+$ are continuous. □

Lemma 3.8. *If the continuous functions $f,g:J\to\mathbb{R}_+$ (defined on the nonvoid open interval $J\subset\mathbb{R}$) satisfy the functional equation* (32) *for every $u,v\in J$, then there exists a constant $c>0$ such that, for every $u\in J$,*

$$f(u)g(u)=c. \tag{34}$$

Proof. If $g(u)=g(v)$ (>0), then, from (32) it follows that $f(u)=f(v)$. Therefore, there exists a function $F:g(J)\to\mathbb{R}_+$ such that

$$f(u)=F\big(g(u)\big)\qquad\text{if}\quad u\in J. \tag{35}$$

If g is constant on J, then f is also constant on J and (34) is evidently satisfied. Therefore, we can assume that g is nonconstant on J. Then, by the continuity of g, $K:=g(J)$ is a nonvoid interval in $\mathbb{R}_+$.

We will show that the function

$$F : K \to \mathbb{R}_+$$

is differentiable on K.

Let $x \in K$ and let $(x_n) \subset K$ be a sequence that converges to x from the left ($x_n < x$) (or from the right $x_n > x$). It is enough to show that

$$\frac{F(x_n) - F(x)}{x_n - x}$$

converges to a limit value depending only on x.

Assume that $x_n < x$ for all $n \in \mathbb{N}$ and let

$$x_0 := \inf\{x_n \mid n \in \mathbb{N}\} = \min\{x_n\}.$$

Then there exist $u_0 \in J$ and $u^* \in J$ such that $g(u_0) = x_0$ and $g(u^*) = x$. We can assume that $u_0 < u^*$ (the other case could be discussed similarly). Let

$$H := \{t \in J \mid u_0 \le t \le u^* \text{ and } g(t) = x\}.$$

Then H is a *closed* set and $H \neq \emptyset$ because of $u^* \in H$. Let

$$u := \inf H,$$

then the inequality $u > u_0$ is clearly satisfied. By the continuity of the function g, we have $g(u) = x$, and if $u_0 \le t < u$, then $g(t) \neq x$. The function g attains every value between x and x_0 on the closed interval $[u_0, u]$, whence, there exists $u_n \in [u_0, u[$ such that $g(u_n) = x_n$ ($n \in \mathbb{N}$). We will show that $u_n \to u$ as $n \to \infty$.

Assume that this is not satisfied. Then there exists a subsequence (u_{n_k}) ($n_1 < n_2 < n_3 < \cdots$) converging to $\bar{u}$ such that $\bar{u} \neq u$ and thus, $\bar{u} < u$. Therefore, by the continuity of g we have $g(u_{n_k}) \to g(\bar{u})$ ($k \to \infty$) and $g(u_{n_k}) = x_{n_k} \to x = g(u)$ ($k \to \infty$). Wherefrom $g(\bar{u}) = g(u)$, which contradicts the definition of u. Thus, indeed, $u_n \to u$ if $n \to \infty$.

By these preliminaries and by equation (32),

$$2f\left(\frac{u_n + u}{2}\right) = \frac{f(u_n)g(u) - f(u)g(u_n)}{g(u) - g(u_n)} = \frac{F(x_n)x - F(x)x_n}{x - x_n}$$

$$= -x\frac{F(x_n) - F(x)}{x_n - x} + F(x).$$

Since f is continuous, we have that

$$\lim_{n\to\infty} f\left(\frac{u_n+u}{2}\right) = f(u) = F[g(u)] = F(x).$$

Thus the limit

$$\lim_{n\to\infty} \frac{F(x_n)-F(x)}{x_n-x} = F'(x)$$

exists and

$$2F(x) = -xF'(x) + F(x),$$

that is,

$$[\ln(xF(x))]' = 0$$

for every $x \in K$. Therefore, there exists $c > 0$ such that

$$F(x) = \frac{c}{x}.$$

This, with (35), yields the statement of the lemma. □

Lemma 3.9. *If the continuous function $f : J \to \mathbb{R}_+$ satisfies the functional equation*

$$\left(f\left(\frac{u+v}{2}\right) - \frac{f(u)+f(v)}{2}\right)\big(f(u)-f(v)\big) = 0 \tag{36}$$

for every $u, v \in J$, then there exist $p, q \in \mathbb{R}$ such that

$$f(u) = pu + q > 0 \quad \text{if } u \in J. \tag{37}$$

Proof. (37) is obviously always a solution of (36) (where, if $p = 0$, then $q > 0$). If f is constant, then there is nothing to prove. Therefore, contrary to the assumption let us suppose that there exists a nonconstant continuous solution $f : J \to \mathbb{R}_+$ of (36) which is not affine, that is, it does not have the form (37). Then there exist $\alpha < \beta$ $(\alpha, \beta \in I)$ such that $f(\alpha) \neq f(\beta)$ and

$$\operatorname{Graph}(f) \not\equiv \operatorname{Graph}(L),$$

where

$$L(u) := \frac{f(\beta)-f(\alpha)}{\beta-\alpha}u - \frac{f(\beta)\alpha - f(\alpha)\beta}{\beta-\alpha} \qquad (u \in J),$$

and

$$\operatorname{Graph} f := \big\{(u, f(u)) \mid u \in J\big\}, \qquad \operatorname{Graph} L := \big\{(u, L(u)) \mid u \in J\big\}.$$

It is clear that

$$(\alpha, f(\alpha)), (\beta, f(\beta)) \in \operatorname{Graph}(f) \cap \operatorname{Graph}(L),$$

thus, since f is continuous, there exist $\alpha^* < \beta^*$ $(\alpha^*, \beta^* \in J)$ such that

$$(\alpha^*, f(\alpha^*)), (\beta^*, f(\beta^*)) \in \operatorname{Graph}(f) \cap \operatorname{Graph}(L),$$

and for every value $t \in]\alpha^*, \beta^*[$ we have $(t, f(t)) \notin \operatorname{Graph}(L)$. For example, we can assume that $f(t) > L(t)$ if $t \in]\alpha^*, \beta^*[$ (the other case can be discussed similarly). In (36) let $u = \alpha^*$, $v = \beta^*$. Then, by $f(\alpha^*) \neq f(\beta^*)$, we obtain

$$\begin{aligned} L(\alpha^*) + L(\beta^*) = f(\alpha^*) + f(\beta^*) &= 2f\left(\frac{\alpha^* + \beta^*}{2}\right) > 2L\left(\frac{\alpha^* + \beta^*}{2}\right) \\ &= L(\alpha^*) + L(\beta^*). \end{aligned}$$

This contradiction completes the proof. □

Now we are ready to prove Theorem 3.6.

Proof of Theorem 3.6. We can assume that $\varphi'(x) > 0$ and $\psi'(x) > 0$ for every $x \in I$. If for example $\varphi'(x) < 0$ $(x \in I)$, then, since $\varphi \sim -\varphi$ on I, we can replace φ by $-\varphi$ to get the desired inequalities. According to Lemma 3.7, $f := \varphi' \circ \varphi^{-1}$ and $g := \psi' \circ \psi^{-1}$ satisfy (32) for every $u, v \in J := \varphi(I)$. By Lemma 3.8, there exists $c > 0$ such that (34) holds for every $u \in J$. Then with the substitution $g(u) = \frac{c}{f(u)}$ from (32), we have that the continuous function $f : J \to \mathbb{R}_+$ satisfies (36); therefore, by Lemma 3.9, there exist $p, q \in \mathbb{R}$ such that

$$f(u) = pu + q > 0, \quad \text{if } u \in J.$$

Thus,

$$\varphi' \circ \varphi^{-1}(u) = pu + q, \quad \text{if } u \in J,$$

wherefrom, with the notation $x = \varphi^{-1}(u) \in I$, we obtain

$$\varphi'(x) = p\varphi(x) + q, \quad \text{if } x \in I. \tag{38}$$

If $p = 0$, then, by $q > 0$ and (38), we have

$$\varphi(x) = qx + r, \quad \text{if } x \in I$$

for some constant $r \in \mathbb{R}$. Hence,

$$\varphi \sim \chi_0 \quad \text{on } I.$$

If $p \neq 0$, then, by (38), we have

$$\varphi(x) = ce^{px} - \frac{q}{p}, \quad \text{if } x \in I$$

for some constant $c \neq 0$, that is,

$$\varphi \sim \chi_p \quad \text{on} \quad I.$$

Finally, it is easy to prove that if $(\varphi, \psi) \in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair and $\varphi \sim \chi_p$ on I, then $\psi \sim \chi_{-p}$ on I. Therefore, the proof is complete. □

3.3. The extension theorem

The previous examinations and results will be strengthened by the *extension theorem* derived in this section. In the introduction we have already mentioned the following very important characteristic that can be deduced from the structure of the functional equation (29). We are now going to discuss it in more details.

Let K be a nonvoid open subinterval of I. If $(\varphi, \psi) \in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair then the restriction $\big(\varphi|_K, \psi|_K\big) \in \mathcal{CM}(K)^2$ is a Matkowski–Sutô pair on the interval K, that is, (29) holds for every $x, y \in K$. In the sequel, instead of $\big(\varphi|_K, \psi|_K\big) \in \mathcal{CM}(K)^2$, we will simply write $(\varphi, \psi) \in \mathcal{CM}(K)^2$.

On the other hand, it is clear that a pair $(\varphi, \psi) \in \mathcal{CM}(I)^2$ on any nonvoid open interval $K \subset I$ is a Matkowski–Sutô pair, if $\varphi \sim \chi_p$ and $\psi \sim \chi_{-p}$ on K for some $p \in \mathbb{R}$. From this, the following very natural question arises: Let $(\varphi, \psi) \in \mathcal{CM}(I)^2$ be a Matkowski–Sutô pair and let $K \subset I$ be a nonvoid open interval. Suppose that

$$\varphi|_K \sim \chi_p, \qquad \psi|_K \sim \chi_{-p} \qquad \text{on} \quad K$$

for some $p \in \mathbb{R}$. Is it true then that the relations $\varphi \sim \chi_p$ and $\psi \sim \chi_{-p}$ hold on I, too? The positive answer to this question generalizes Theorems 3.3 and 3.5 in the following forms.

Theorem 3.10. *If $(\varphi,\psi)\in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair and there exists a nonvoid open interval $K\subset I$ such that one of the restrictions of φ and ψ to K is continuously differentiable on K, then there exists $p\in\mathbb{R}$ such that* (30) *holds.*

Theorem 3.11. *If $(\varphi,\psi)\in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair and there exists a nonvoid open interval $K\subset I$ such that the quasi-arithmetic means $M=A_\varphi$ and $N=A_\psi$ are strictly comparable in K, then there exists $p\in\mathbb{R}$ such that* (30) *holds.*

To prove the extension theorem, we need the following lemmas.

Lemma 3.12. *If $(\varphi,\psi)\in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair and there exists a nonvoid open interval $K\subset I$ such that $\varphi\sim\chi_p$ and $\psi\sim\chi_{-p}$ on K for some $p\in\mathbb{R}$, then there exists a Matkowski–Sutô pair $(\widetilde{\varphi},\widetilde{\psi})\in \mathcal{CM}(I)^2$ such that $\varphi\sim\widetilde{\varphi}$ and $\psi\sim\widetilde{\psi}$ on I, and*

$$\widetilde{\varphi}(x)=\chi_p(x),\quad \widetilde{\psi}(x)=\chi_{-p}(x)\quad \text{if } x\in K.$$

Proof. There exist constants $A_i\neq 0$ and B_i $(i=1,2)$ such that

$$\varphi(x)=A_1\chi_p(x)+B_1,\qquad \psi(x)=A_2\chi_{-p}(x)+B_2$$

for every $x\in K$ for some $p\in\mathbb{R}$. Let

$$\widetilde{\varphi}(x):=\frac{1}{A_1}\varphi(x)-\frac{B_1}{A_1}\quad\text{and}\quad \widetilde{\psi}(x):=\frac{1}{A_2}\psi(x)-\frac{B_2}{A_2}\quad\text{if } x\in I.$$

Then, by $\varphi\sim\widetilde{\varphi}$ and $\psi\sim\widetilde{\psi}$, we have that $(\widetilde{\varphi},\widetilde{\psi})\in\mathcal{CM}(I)^2$ is a Matkowski–Sutô pair and, for every $x\in K$,

$$\widetilde{\varphi}(x)=\frac{1}{A_1}\big(A_1\chi_p(x)+B_1\big)-\frac{B_1}{A_1}=\chi_p(x),$$
$$\widetilde{\psi}(x)=\frac{1}{A_2}\big(A_2\chi_{-p}(x)+B_2\big)-\frac{B_2}{A_2}=\chi_{-p}(x).$$ □

Lemma 3.13. *Let $\varphi:[A,B]\to\mathbb{R}$ $(A<B)$ be a continuous and strictly monotone increasing function that satisfies*

$$\tau:=\frac{B-A}{\varphi(B)-\varphi(A)}\geq 1. \tag{39}$$

Furthermore, suppose that the function

$$f(t) := \varphi(t) - t \qquad (t \in [A, B]) \tag{40}$$

satisfies the functional equation

$$f(x + y - A_\varphi(x, y)) = f(A_\varphi(x, y)) \tag{41}$$

for every $x, y \in [A, B]$. Then there exists $\sigma \in \mathbb{R}$ such that

$$\varphi(x) = \frac{1}{\tau}\, x + \sigma \tag{42}$$

holds for every $x \in [A, B]$.

Pproof. We note that the functional equation (41) is in fact an equation with respect to the unknown function φ.

Let $\varphi([A, B]) = [\alpha, \beta]$ where $\varphi(A) = \alpha$ and $\varphi(B) = \beta$, and since φ is strictly increasing therefore, $\alpha < \beta$. Furthermore, let

$$g(u) := \varphi^{-1}(u) \quad \text{if } u \in [\alpha, \beta]. \tag{43}$$

Then, by (41), with the substitutions $x := \varphi^{-1}(u)$, $y := \varphi^{-1}(v)$, we get

$$\varphi\left(\varphi^{-1}(u) + \varphi^{-1}(v) - \varphi^{-1}\left(\frac{u+v}{2}\right)\right) - \varphi^{-1}(u) - \varphi^{-1}(v)$$
$$+\varphi^{-1}\left(\frac{u+v}{2}\right) = \frac{u+v}{2} - \varphi^{-1}\left(\frac{u+v}{2}\right)$$

for every $u, v \in [\alpha, \beta]$, whence

$$g(u)+g(v)-g\left(\frac{u+v}{2}\right) = g\left(g(u)+g(v)-2g\left(\frac{u+v}{2}\right)+\frac{u+v}{2}\right). \tag{44}$$

Now let

$$b(u) := \frac{B-A}{\beta-\alpha}u + \frac{A\beta - B\alpha}{\beta-\alpha} - g(u) \tag{45}$$

if $u \in [\alpha, \beta]$. It is easy to see that

$$b(\alpha) = b(\beta) = 0 \tag{46}$$

and, by (3)

$$\tau := \frac{B-A}{\beta-\alpha} \geq 1. \tag{47}$$

Hence, (45) implies $g(u) = \tau u + \eta - b(u)$, where $\eta := \frac{A\beta - B\alpha}{\beta-\alpha}$, and therefore, from (44) we obtain that

$$\begin{aligned}
&\tau u + \eta - b(u) + \tau v + \eta - b(v) - \tau\frac{u+v}{2} - \eta + b\Big(\frac{u+v}{2}\Big) \\
&= g\Big(\tau u + \eta - b(u) + \tau v + \eta - b(v) - 2\Big(\tau\frac{u+v}{2} + \eta - b\Big(\frac{u+v}{2}\Big)\Big) \\
&\qquad + \frac{u+v}{2}\Big) = g\Big(2b\Big(\frac{u+v}{2}\Big) - b(u) - b(v) + \frac{u+v}{2}\Big)
\end{aligned}$$

follows for every $u, v \in [\alpha, \beta]$. Rearranging this equation, we have

$$\begin{aligned}
&b\left(2b\left(\frac{u+v}{2}\right) - b(u) - b(v) + \frac{u+v}{2}\right) \\
&\qquad = (1-\tau)b(u) + (1-\tau)b(v) + (2\tau-1)b\left(\frac{u+v}{2}\right)
\end{aligned} \tag{48}$$

for all $u, v \in [\alpha, \beta]$, where, by (47), $\tau \geq 1$. The unknown function $b : [\alpha, \beta] \to \mathbb{R}$ is continuous and satisfies (46) and (48). We will show that $b(u) = 0$ for every $u \in [\alpha, \beta]$.

On the contrary, assume that there exists $v_0 \in]\alpha, \beta[$ such that $b(v_0) \neq 0$. Then there are two possible cases : (i) $b(v_0) > 0$ or (ii) $b(v_0) < 0$.

In case (i) let

$$\max_{u\in[\alpha,\beta]} b(u) = M > 0$$

and

$$S := \big\{u \mid u \in [\alpha, \beta],\ b(u) = M\big\}, \quad u_0 := \sup S.$$

Then $b(u_0) = M$ and $u_0 < \beta$. Thus, there exists $\varepsilon > 0$ such that $u_0 - \varepsilon, u_0 + \varepsilon \in]\alpha, \beta[$. Substituting the values $u := u_0 - \varepsilon$ and $v := u_0 + \varepsilon$ into (48), we obtain the equation

$$\begin{aligned}
&b\big(2M - b(u_0-\varepsilon) - b(u_0+\varepsilon) + u_0\big) \\
&\qquad = (1-\tau)b(u_0-\varepsilon) + (1-\tau)b(u_0+\varepsilon) + (2\tau-1)M.
\end{aligned}$$

By the definition of the number u_0,

$$v(\varepsilon) := 2M - b(u_0 - \varepsilon) - b(u_0 + \varepsilon) > 0,$$

and from the previous equation we have

$$\begin{aligned} b\big(v(\varepsilon) + u_0\big) &= (1-\tau)(2M - v(\varepsilon)) + (2\tau - 1)M \\ &= M + (\tau - 1)v(\varepsilon). \end{aligned} \tag{49}$$

For $\tau > 1$ the equation (49) leads to a contradiction, because the maximum value of the function b is M and $v(\varepsilon) > 0$. If $\tau = 1$, then, by (49), $u_0 + v(\varepsilon) \in S$ and $u_0 + v(\varepsilon) > u_0$, which contradicts the definition of u_0. Thus, case (i) is not possible.

If (ii) holds, then let

$$\min_{u \in [\alpha, \beta]} b(u) = m < 0$$

and

$$T := \big\{u \mid u \in [\alpha, \beta],\ b(u) = m\big\}, \quad u_0 := \inf T.$$

Then $b(u_0) = m$ and $\alpha < u_0$. Whence, there exists $\varepsilon > 0$ such that $u_0 - \varepsilon, u_0 + \varepsilon \in]\alpha, \beta[$. Substituting the values $u := u_0 - \varepsilon$ and $v := u_0 + \varepsilon$ into (48), we get

$$b\big(2m - b(u_0 - \varepsilon) - b(u_0 + \varepsilon) + u_0\big) = (1-\tau)\big(b(u_0 - \varepsilon) + b(u_0 + \varepsilon)\big) + (2\tau - 1)m.$$

Then, by the definition of the number u_0, we have

$$v(\varepsilon) := 2m - b(u_0 - \varepsilon) - b(u_0 + \varepsilon) < 0,$$

and thus, from the previous equation, we obtain

$$b\big(u_0 + v(\varepsilon)\big) = m + (1 - \tau)v(\varepsilon).$$

If $\tau > 1$, then, from the previous equation, we would obtain that b has smaller values than m, which is not possible. If $\tau = 1$, then, by $v(\varepsilon) < 0$, $u_0 + v(\varepsilon) < u_0$, and in this point b has the value m, which contradicts the definition of u_0. Therefore, case (ii) also leads to a contradiction.

Whence, we proved that $b(u) = 0$ for every $u \in [\alpha, \beta]$. Therefore, by (45), (47) and (43), we have

$$\varphi^{-1}(u) = g(u) = \tau u + \eta, \qquad \text{if } u \in [\alpha, \beta],$$

thus, by the notation $\sigma := -\frac{\eta}{\tau}$

$$\varphi(x) = \frac{1}{\tau}x + \sigma, \quad \text{if } x \in [A, B],$$

that is, (42) holds. Hence, our proof is complete. □

Now we are ready to state the extension theorem ([16]).

Theorem 3.14. *If $(\varphi, \psi) \in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair and there exist a nonvoid open interval $K \subset I$ and $p \in \mathbb{R}$ such that $\varphi \sim \chi_p$ and $\psi \sim \chi_{-p}$ on K, then $\varphi \sim \chi_p$ and $\psi \sim \chi_{-p}$ on I.*

Proof. Because of Lemma 3.12, we can assume that

$$\varphi(x) = \chi_p(x) \quad \textit{and} \quad \psi(x) = \chi_{-p}(x) \quad \text{if } x \in K. \tag{50}$$

Moreover, we can also assume that the open interval K is maximal, that is, there is no strictly larger interval where the above equalities are satisfied. We are going to show that then K must be identical with I.

Let $K =]a, b[$ and suppose that $K \neq I$. Then either a or b or both of them are elements of I. We only consider the case when $a \in I$, the other cases can be handled similarly. Choose a number b^* with $a < b^* < b$.

Because of the continuity and strict monotonicity, there exists $0 < \delta < b - a$ such that

$$\frac{\varphi(x) + \varphi(y)}{2} \in \varphi(K) \quad \textit{and} \quad \frac{\psi(x) + \psi(y)}{2} \in \psi(K) \tag{51}$$

hold for every $x \in [a - \delta, a] \subset I$, and $y \in]b^* - \delta, b^*[\subset I$.

Now there are two possible cases: (i) $p \neq 0$ and (ii) $p = 0$.

The Case (i) can be handled relatively easily. Then (50) implies

$$\varphi^{-1}(t) = \frac{1}{p}\log t, \quad \text{if } t \in \varphi(K) \subset \mathbb{R}_+ \tag{52}$$

and

$$\psi^{-1}(t) = \frac{1}{p}\log t, \quad \text{if } t \in \psi(K) \subset \mathbb{R}_+. \tag{53}$$

Using (50), (52), and (53), by (51), from the functional equation (29), we obtain

$$\frac{1}{p}\log\frac{\varphi(x) + e^{py}}{2} - \frac{1}{p}\log\frac{\psi(x) + e^{-py}}{2} = x + y \tag{54}$$

for every $x \in [a-\delta, a]$, and $y \in]b^*-\delta, b^*[$. From (54) it follows that

$$e^{py}\big(e^{px}\psi(x)-1\big) = \varphi(x) - e^{px},$$

wherefrom

$$e^{px}\psi(x) - 1 = 0$$

for every $x \in [a-\delta, a]$, that is, $\psi(x) = e^{-px}$ and $\varphi(x) = e^{px}$ if $x \in [a-\delta, a]$. This means that the solutions of the form (50) can be extended from $K =]a, b[$ to the open interval $K_1 :=]a-\delta, b[\subset I$, which properly contains K contradicting the maximality of K. This completes the proof in the Case (i).

The investigation of Case (ii) is much more difficult. In this case (50) implies

$$\varphi(x) = \psi(x) = x \quad \text{if } x \in K =]a, b[. \tag{55}$$

Then the roles of the functions φ and ψ can be interchanged, and since $a \in I$, by the continuity,

$$\varphi(a) = \psi(a) = a. \tag{56}$$

On the other hand, for every $x \in [a-\delta, a]$ and $y \in [b^*-\delta, b^*[$, (51), (29), and (55) yield that

$$\frac{\varphi(x)+y}{2} + \frac{\psi(x)+y}{2} = x + y$$

that is,

$$\varphi(x) + \psi(x) = 2x, \quad \text{if } x \in K = [a-\delta, a]. \tag{57}$$

The functions φ and ψ are *strictly increasing* on I, thus, also on the interval $[a-\delta, a]$. Let $\varphi(a-\delta) =: c$ and $\psi(a-\delta) =: C$. Then, by (57), $c + C = 2(a-\delta)$, thus, either c or C is greater than or equal to $a-\delta$. Since the roles of φ and ψ can be interchanged, we can assume that

$$\varphi(a-\delta) = c \geq a-\delta. \tag{58}$$

Now let $x, y \in [a-\delta, a]$ arbitrary. Then, by (57) and the Matkowski–Sutô equation (29), we have

$$\psi^{-1}\left(\frac{2x - \varphi(x) + 2y - \varphi(y)}{2}\right) = x + y - A_\varphi(x, y),$$

from which, applying (57) again, we obtain

$$x+y-\frac{\varphi(x)+\varphi(y)}{2}=2x+2y-2A_\varphi(x,y)-\varphi\big(x+y-A_\varphi(x,y)\big)$$

for every $x,y\in[a-\delta,a]$. Rearranging the above equation, and defining the function

$$f(t):=\varphi(t)-t\quad\text{if } t\in[a-\delta,a],$$

we obtain

$$f\big(x+y-A_\varphi(x,y)\big)=f\big(A_\varphi(x,y)\big) \tag{59}$$

for every $x,y\in[a-\delta,a]$. Due to (56) and (58), we have

$$\tau:=\frac{a-(a-\delta)}{\varphi(a)-\varphi(a-\delta)}=\frac{\delta}{a-c}\geq 1.$$

Hence, the strictly increasing and continuous function φ satisfies the conditions of Lemma 3.13 on the closed interval $[a-\delta,a]=:[A,B]$, therefore, we have

$$\varphi(x)=\frac{1}{\tau}x+\sigma\quad\text{if } x\in[a-\delta,a], \tag{60}$$

where $\sigma:=a-\frac{a}{\tau}$. Form this and (57), we obtain

$$\psi(x)=2x-\varphi(x)=\left(2-\frac{1}{\tau}\right)x-\sigma,\quad\text{if } x\in[a-\delta,a].$$

Now let $x\in[a-\delta,a]$ and $y\in[a,b^*]$ satisfy

$$\frac{\varphi(x)+\varphi(y)}{2}\in\varphi([a-\delta,a])\quad\text{and}\quad\frac{\psi(x)+\psi(y)}{2}\in\psi([a-\delta,a]).$$

Then, by

$$\varphi^{-1}(s)=\tau(s-\sigma)\quad\text{and}\quad\psi^{-1}(s)=\frac{\tau}{2\tau-1}(s+\sigma)$$

and the Matkowski–Sutô' equation (29), we get

$$\tau\left(\frac{\frac{1}{\tau}x+\sigma+y}{2}-\sigma\right)+\frac{\tau}{2\tau-1}\left(\frac{\left(2-\frac{1}{\tau}\right)x-\sigma+y}{2}+\sigma\right)=x+y,$$

which implies

$$\left(\tau + \frac{\tau}{2\tau - 1}\right) y + \left(\frac{\tau}{2\tau - 1} - \tau\right) \sigma = 2y.$$

Since the last equation holds for every y from some interval, therefore,

$$\tau + \frac{\tau}{2\tau - 1} = 2$$

and so necessarily $\tau = 1$ and $\sigma = 0$. Hence,

$$\varphi(x) = \psi(x) = x \quad \text{if } x \in [a - \delta, a],$$

that is, the solutions can be extended from $K =]a, b[$ to $K_1 =]a - \delta, b[$, where K is a proper subset of K_1. The contradiction obtained completes our proof. □

4. The solution of the Matkowski–Sutô problem

4.1. Regularity of functional equations and Hilbert's fifth problem

The title of Hilbert's fifth problem ([24]) is the following: *Lie's concept of a continuous group of transformations without the assumption of the differentiability of the functions defining the group.* Hilbert refers to the generality of the problem when he states that the functions defining a continuous group of transformations satisfy such functional equations for which the differentiability of the unknown functions need not be assumed because their continuity seems to be sufficient.

In our opinion the Matkowski–Sutô problem is a special case of Hilbert's fifth problem, because it can be reformulated in the following form.

Let $X \subset \mathbb{R}$ be a nonvoid open interval on which three *two variable operations* $\circ_i : X^2 \to X$ $(i = 1, 2, 3)$ are defined with the property $\mathcal{QA}$. We say that an operation $\circ : X^2 \to X$ has the property $\mathcal{QA}$ if it satisfies the following conditions:

(QA1) $\circ$: $X^2 \to X$ is continuous;

(QA2) $\circ$ is commutative (i.e., $x \circ y = y \circ x$ for all $x, y \in X$);

(QA3) $\circ$ is reflexive (i.e., $x \circ x = x$ for all $x \in X$);

(QA4) $\circ$ is cancellative (i.e., if $x \circ y = x \circ z$, then $y = z$ for all $x, y, z \in X$);

(QA5) $\circ$ is bisymmetric (i.e., $(x \circ y) \circ (u \circ v) = (x \circ u) \circ (y \circ v)$ for all $x, y, u, v \in X$).

Now we can reformulate the Matkowski–Sutô problem in the following form. *Find those binary operations $\circ_1$, $\circ_2$, and $\circ_3$ defined on X with the property $\mathcal{QA}$ which satisfy the following equation*

$$a \circ_1 b = (a \circ_2 b) \circ_1 (a \circ_3 b) \tag{61}$$

for every $a, b \in X$.

According to Aczél's theorem ([2]), an operation $\circ : X^2 \to X$ has the property $\mathcal{QA}$ if and only if there exists $f \in \mathcal{CM}(X)$ such that

$$a \circ b = f^{-1}\left(\frac{f(a) + f(b)}{2}\right) = A_f(a, b)$$

for every $a, b \in X$, that is, $\circ : X^2 \to X$ is a quasi-arithmetic mean. From this, we have that the "invariance" equation (61) means the following. Since $\circ_i$ is a quasi-arithmetic mean for every i, there exist generating functions $f_1, f_2, f_3 \in \mathcal{CM}(X)$ such that

$$A_{f_1}(a, b) = A_{f_1}\big(A_{f_2}(a, b), A_{f_3}(a, b)\big) \tag{62}$$

holds for every $a, b \in X$, that is, $A_{f_1} = A_{f_2} \otimes A_{f_3}$. From (62), we obtain

$$f_1(a) + f_1(b) = f_1\big(A_{f_2}(a, b)\big) + f_1\big(A_{f_3}(a, b)\big)$$

for every $a, b \in X$; wherefrom, with the substitutions

$$x := f_1(a), \qquad y := f_1(b), \qquad x, y \in f_1(X) =: I,$$

and with the notations

$$\varphi := f_2 \circ f_1^{-1}, \quad \psi := f_3 \circ f_1^{-1},$$

we have that I is a nonvoid open interval and $\varphi, \psi \in \mathcal{CM}(I)$ satisfy the Matkowski–Sutô equation (29). So the solution of problem (61) reduces to the problem of solving (29), in which it is unnatural to make further assumptions for the unknown functions φ, ψ according to Hilbert's stipulation. Therefore, it is of theoretical importance to study the solutions of the functional equation (29) under the natural condition $\varphi, \psi \in \mathcal{CM}(I)$.

According to our preceding investigations, the Matkowski–Sutô problem is solved if there exists a nonvoid open interval $K \subset I$, in which the Matkowski–Sutô pair $(\varphi, \psi) \in \mathcal{CM}(I)^2$ is continuously differentiable and their derivatives do not vanish in K.

Therefore, if we are able to show that the previous *regularity* property holds for any Matkowski–Sutô pair $(\varphi, \psi) \in \mathcal{CM}(I)^2$, then this will lead to the complete solution of the problem. This is the subject of the following sections, and the result obtained will verify Hilbert's fifth problem in this case, too.

4.2. Locally Lipschitz property of Matkowski–Sutôpairs

If $(\varphi, \psi) \in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair, then, from the Matkowski–Sutô equation (29), with the notations $J := \varphi(I)$ and $u = \varphi(x)$, $v = \varphi(y)$ $(u, v \in J)$, we deduce that

$$\begin{aligned} &\psi^{-1}\left(\frac{\psi \circ \varphi^{-1}(u) + \psi \circ \varphi^{-1}(v)}{2}\right) \\ &= \varphi^{-1}(u) + \varphi^{-1}(v) - \varphi^{-1}\left(\frac{u+v}{2}\right) \end{aligned} \tag{63}$$

holds for every $u, v \in J$. Without loss of generality, we may assume that the functions $(\varphi, \psi) \in \mathcal{CM}(I)^2$ are strictly monotone *increasing* on I because, in the decreasing case, they can be substituted by equivalent and increasing generating functions. Then φ, ψ are strictly monotone increasing on I, and φ^{-1}, ψ^{-1} are also strictly monotone increasing on $J := \varphi(I)$, and on $L := \psi(I)$, where J and L are nonvoid open intervals in $\mathbb{R}$. From these properties, one derives that the left side of (63), namely the function

$$u \mapsto \psi^{-1}\left(\frac{\psi \circ \varphi^{-1}(u) + \psi \circ \varphi^{-1}(v)}{2}\right) \qquad (u \in J)$$

strictly increases on J for any fixed $v \in J$. Therefore, the right side of (63) is also strictly increasing in u, that is, the function

$$u \mapsto \varphi^{-1}(u) - \varphi^{-1}\left(\frac{u+v}{2}\right) \qquad (u \in J) \tag{64}$$

is *strictly increasing* on J *for every* $v \in J$. Of course, φ^{-1} is also *continuous* on J. This bears some *further* information about the function φ^{-1} (and similarly about ψ^{-1}). Therefore, the following theorem is fundamental.

Theorem 4.1. *Let $J \subset \mathbb{R}$ be a nonvoid open interval and let $f : J \to \mathbb{R}$ be a strictly increasing and continuous function such that, for any $v \in J$, the function*

$$u \mapsto f(u) - f\left(\frac{u+v}{2}\right) \qquad (u \in J) \tag{65}$$

is strictly increasing.

Then, for every $u_0 \in J$, there exist $\delta > 0$, and $K, L > 0$ such that $U :=]u_0 - \delta, u_0 + \delta[\subset J$, and for all $u, v \in U$, $u \neq v$, we have

$$0 < K \leq \frac{f(u) - f(v)}{u - v} \leq L. \tag{66}$$

Proof. (i) First, we prove the existence of the upper bound L.

Due to the condition of the theorem, if $u \leq u'$ $(u, u' \in J)$, then

$$f(u) - f\left(\frac{u+v}{2}\right) \leq f(u') - f\left(\frac{u'+v}{2}\right),$$

that is,

$$0 \leq f\left(\frac{u'+v}{2}\right) - f\left(\frac{u+v}{2}\right) \leq f(u') - f(u).$$

Hence

$$\left| f\left(\frac{u'+v}{2}\right) - f\left(\frac{u+v}{2}\right) \right| \leq |f(u') - f(u)| \tag{67}$$

for all $u, u', v \in J$.

Let $u_0 \in J$ be arbitrary. Then there exists $r > 0$ such that $]u_0 - r, u_0 + r[\subset J$. Since f is monotone, according to Lebesgue's theorem, there exists a point $u^* \in]u_0 - \frac{r}{2}, u_0 + \frac{r}{2}[$ at which the function f is differentiable. Thus, there exists $0 < \varrho < \frac{r}{2}$ such that if $|u - u^*| < \varrho$ and $u, u^* \in J$, $u \neq u^*$, then

$$\left| \frac{f(u) - f(u^*)}{u - u^*} - f'(u^*) \right| \leq 1,$$

that is,

$$|f(u) - f(u^*)| \leq \big(|f'(u^*)| + 1\big)|u - u^*| = L^*|u - u^*|, \tag{68}$$

where $L^* := |f'(u^*)| + 1$. It is clear that (68) holds for every $|u - u^*| < \varrho$.

Now let $x, y \in]u_0 - \frac{\varrho}{4}, u_0 + \frac{\varrho}{4}[\subset J$ be arbitrary. Then, with the notations $v := 2y - u^*$ and $u := 2x - v = 2(x - y) + u^*$, we obtain

$$|v - u_0| = |2y - u^* - u_0| \leq |2(y - u_0)| + |u_0 - u^*| < 2\frac{\varrho}{4} + \frac{r}{2} = \frac{\varrho}{2} + \frac{r}{2} < r,$$

thus, $v \in J$ and

$$|u - u^*| = |2(x - y)| < 2\frac{\varrho}{2} = \varrho.$$

From this, by (67) and (68), we have

$$\begin{aligned} |f(x) - f(y)| &= \left| f\left(\frac{u+v}{2}\right) - f\left(\frac{u^*+v}{2}\right)\right| \\ &\leq |f(u) - f(u^*)| \leq L^*|u - u^*| = 2L^*|x - y|, \end{aligned}$$

that is, with the notations $\delta := \frac{\varrho}{4}$ and $L := 2L^*$, we have proven the right hand side inequality of (66).

(ii) In order to prove the left hand side inequality of (66), we shall make use of the following observation. Since $f : J \to \mathbb{R}$ is strictly monotone increasing, therefore, f' exists on a dense subset of J and it is positive. Indeed, if f' were zero almost everywhere on a nonvoid open interval $H \subset J$, then, according to what we have proved in part (i), f is absolutely continuous, hence we would have

$$f(y) - f(x) = \int_x^y f'(t)dt = 0 \qquad (x, y \in H),$$

contradicting the strict monotonicity of f.

Thus, let $u_0 \in J$ and $r > 0$ be such that $]u_0 - r, u_0 + r[\subset J$. Let $u^* \in]u_0 - \frac{r}{3}, u_0 + \frac{r}{3}[$ be such a point at which f is differentiable and

$$f'(u^*) > 0.$$

Then there exists $0 < \varrho < \frac{r}{3}$ such that, if $|u - u^*| < \varrho$ $(u \neq u^*)$, then

$$\left| \frac{f(u) - f(u^*)}{u - u^*} - f'(u^*) \right| \leq \frac{f'(u^*)}{2}.$$

From here, we obtain that

$$|f(u) - f(u^*)| \geq \frac{f'(u^*)}{2}|u - u^*| \tag{69}$$

for all $|u-u^*|<\varrho$.

Now let $u,u'\in\,]u_0-\varrho,u_0+\varrho[$ and $v:=2u^*-u$. Then

$$|v-u_0|=|2u^*-u-u_0|=|2(u^*-u_0)+u_0-u|\le 2|u^*-u_0|$$
$$+|u_0-u|\le 2\frac{r}{3}+\varrho<r,$$

whence, $v\in J$. Thus, by (67) and (69),

$$\left|\frac{u'-u+2u^*}{2}-u^*\right|=\frac{|u'-u|}{2}<\frac{2\varrho}{2}=\varrho$$

which implies that

$$|f(u')-f(u)|\ge\left|f\left(\frac{u'+v}{2}\right)-f\left(\frac{u+v}{2}\right)\right|=\left|f\left(\frac{u'-u+2u^*}{2}\right)-f(u^*)\right|$$
$$\ge\frac{f'(u^*)}{2}\left|\frac{u'-u+2u^*}{2}-u^*\right|=\frac{f'(u^*)}{4}\,|u'-u|.$$

Therefore, with the choices $\delta:=\varrho$ and $K:=\frac{f'(u^*)}{4}$, we can see that the left hand side inequality of (66) holds, too. □

Definition 4.2. Let $J\subset\mathbb{R}$ be a nonvoid open interval and $f:J\to\mathbb{R}$. The function f is said to satisfy the *local Lipschitz* condition on its domain if, for every $u_0\in J$, there exist $\delta>0$ and $L>0$ constant such that $U:=\,]u_0-\delta,u_0+\delta[\,\subset J$ and, for all $u,v\in U$,

$$|f(u)-f(v)|\le L|u-v|. \tag{70}$$

Theorem 4.3. *If $(\varphi,\psi)\in\mathcal{CM}(I)^2$ is a Matkowski–Sutô pair, then φ, ψ, φ^{-1}, ψ^{-1} are locally Lipschitz functions on their domains.*

Proof. If φ and ψ are increasing, then, by (64), $\varphi^{-1}:J\to\mathbb{R}$ satisfies the conditions of Theorem 4.1. Thus, for all $u_0\in J$, there exist $\delta>0$ and $K,L>0$ such that $U:=\,]u_0-\delta,u_0+\delta[\,\subset J$, and, for any $u,v\in U$, $u\ne v$,

$$0<K\le\frac{\varphi^{-1}(u)-\varphi^{-1}(v)}{u-v}\le L.$$

From the right hand side of this inequality, we have that the function $\varphi^{-1}:J\to I$ satisfies the local Lipschitz condition with the Lipschitz

constant L. From the left hand side of the inequality, we can deduce that the function $\varphi : I \to J$ also satisfies the local Lipschitz condition with the Lipschitz constant $\frac{1}{K} > 0$. Since the roles of φ and ψ are interchangeable, we obtain that functions ψ^{-1} and ψ both have the local Lipschitz condition on their interval domains.

If either φ or ψ is decreasing, then, replacing them by equivalent strictly increasing generators, we have that they and their inverses are also locally Lipschitz functions. □

Corollary 4.4. *If $(\varphi, \psi) \in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair, then in all those points $x_0 \in I$ where φ (or ψ) is differentiable, it must be true that $\varphi'(x_0) \neq 0$ (or $\psi'(x_0) \neq 0$).*

Proof. If φ and ψ are increasing, then there exist $\delta > 0$ and $K > 0$ such that $U :=]u_0 - \delta, u_0 + \delta[\subset J$, and for all $x, y \in U$, $x \neq y$

$$0 < K \leq \frac{\varphi(x) - \varphi(y)}{x - y}.$$

Thus, with the choice $y = x_0$, we obtain

$$0 < K \leq \lim_{x \to x_0} \frac{\varphi(x) - \varphi(x_0)}{x - x_0} = \varphi'(x_0).$$

For decreasing φ the proof is similar. □

4.3. Differentiability of Matkowski–Sutô pairs

If $(\varphi, \psi) \in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair, then, by the result of the previous paragraph, the functions φ, ψ, φ^{-1} and ψ^{-1} satisfy a local Lipschitz condition on their domains. It follows from this that the function $h := \psi \circ \varphi^{-1}$ is also locally Lipschitz on interval $J := \varphi(I)$. From this we have that $h \in \mathcal{CM}(I)$ "keeps null sets", that is, for all measurable null sets $E \subset I$ the image $h(E)$ has zero measure (Natanson, [44]). This property plays an important role in the following examinations.

Definition 4.5. Let $f : J \to \mathbb{R}$ ($J \subset \mathbb{R}$ is a nonvoid open interval) be an arbitrary function. We say that $t \in J$ is a *point of symmetry* for f, in notation $t \in \sigma(f)$ if, for all $s \in J_t := (J - t) \cap (t - J)$, we have

$$f(t + s) + f(t - s) = 2f(t). \tag{71}$$

Lemma 4.6. *If $f : J \to \mathbb{R}$ is a continuous function, then the set of all points of symmetry for f is a closed set, that is, $\sigma(f)$ is closed in J.*

Proof. Let t_n be a sequence in $\sigma(f)$ converging to $t \in J$. We intend to show that then $t \in \sigma(f)$ holds, too.

Let $s \in J_t = (J - t) \cup (t - J)$. Then, due to the openness of J, we have that $s \in (J - t_n) \cup (t_n - J) = J_{t_n}$ for all large values of n. Thus,

$$f(t_n + s) + f(t_n - s) = 2f(t_n).$$

Taking the limit $n \to \infty$ and using the continuity of f, we get that (71) holds, i.e., t belongs to $\sigma(f)$. □

The following result shows that, except at points of symmetry, the solutions φ, ψ of the Matkowski–Sutô equation are differentiable.

Theorem 4.7. *If $(\varphi, \psi) \in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair, and $t_0 \in J := \varphi(I)$ is not a point of symmetry for φ^{-1}, that is, $t_0 \notin \sigma(\varphi^{-1})$, then φ^{-1} is differentiable at t_0.*

Proof. With the substitutions $x := \varphi^{-1}(t + s)$, $y := \varphi^{-1}(t - s)$ for any $t \in J$ and $s \in J_t := (J - t) \cap (t - J)$, by equation (29), we have that

$$\varphi^{-1}(t) = \varphi^{-1}(t + s) + \varphi^{-1}(t - s) - \psi^{-1}\left(\frac{h(t + s) + h(t - s)}{2}\right), \tag{72}$$

where $h := \psi \circ \varphi^{-1}$. If $t \in \sigma(h)$, then, by (72), $t \in \sigma(\varphi^{-1})$. On the other hand, if $t \in \sigma(\varphi^{-1})$, then, by (72), $t \in \sigma(h)$. Whence,

$$\sigma(h) = \sigma(\varphi^{-1}).$$

Now let $t_0 \notin \sigma(\varphi^{-1})$. For any function $g : J_{t_0} \to \mathbb{R}$, denote by N_g the set of all those points $s \in J_{t_0}$ in which g is *not differentiable*. For $s \in J_{t_0}$ let us define the following functions

$$\begin{aligned} g_1(s) &:= \varphi^{-1}(t_0 + s), \\ g_2(s) &:= \varphi^{-1}(t_0 - s), \\ g_3(s) &:= h(t_0 + s), \\ g_4(s) &:= h(t_0 - s). \end{aligned}$$

Since the functions g_i $(i = 1, 2, 3, 4)$ are monotone, according to Lebesgue's theorem, N_{g_i} $(i = 1, 2, 3, 4)$ is a null set, that is, the set

$$N := \bigcup_{i=1}^{4} N_{g_i} \subset J_{t_0}$$

has zero Lebesgue measure. Since $t_0 \notin \sigma(\varphi^{-1}) = \sigma(h)$ thus, the function

$$h_{t_0}(s) := \frac{h(t_0 + s) + h(t_0 - s)}{2} \qquad (s \in J_{t_0})$$

is nonconstant; therefore, its image $H_0 := h_{t_0}(J_{t_0})$ is a *proper interval*. Let

$$C := \big\{u \in H_0 \mid \psi^{-1} \text{ is not differentiable at } u\big\}.$$

Then, by Lebesgue's theorem, C is a null set, whence, the set $H_0 \backslash C$ has positive measure.

Let

$$D := h_{t_0}^{-1}(H_0 \setminus C) \subseteq J_{t_0}.$$

Then $h_{t_0}(D) = H_0 \setminus C$. If D were a null set, then $h_{t_0}(D)$ would also be a null set, since h_{t_0} is a locally Lipschitz function due to Theorem 4.3. Hence, $D \subset I_{t_0}$ has a positive measure. Thus $D \setminus N$ also has a positive measure, hence $D \setminus N$ is nonvoid, i.e., there exists

$$s_0 \in D \setminus N.$$

Then g_1, g_2, g_3 and g_4 are differentiable at s_0 and ψ^{-1} is differentiable at $h_{t_0}(s_0)$. According to (72),

$$\varphi^{-1}(t) = \varphi^{-1}(t + s_0) + \varphi^{-1}(t - s_0) - \psi^{-1}\left(\frac{h(t + s_0) + h(t - s_0)}{2}\right) \tag{73}$$

for every $t \in J$, furthermore, φ^{-1} is differentiable at the points $(t_0 + s_0)$ and $(t_0 - s_0)$, h is differentiable at the points $(t_0 + s_0)$ and $(t_0 - s_0)$, and ψ^{-1} is differentiable at $h_{t_0}(s_0)$. Thus, we have that the right hand side of (72) is differentiable at t_0 because of the chain rule. Therefore, by (72), φ^{-1} is differentiable at t_0. Hence, the proof of the theorem is complete. □

Now we are going to prove the following important theorem.

Theorem 4.8. *If $(\varphi,\psi)\in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair, then there exists a nonvoid open interval $K\subset I$ on which φ and ψ are differentiable and $\varphi'(x)\neq 0$, $\psi'(x)\neq 0$ if $x\in K$.*

Proof. Let us consider the function $\varphi^{-1}: J\to I$, where $J:=\varphi(I)$. Then there are two possible cases:

(i) $\sigma(\varphi^{-1})=J$, that is, every $t\in J$ is a point of symmetry for φ^{-1}; or

(ii) $\sigma(\varphi^{-1})\neq J$, that is, φ^{-1} has a point of non-symmetry in J.

In Case (i), for all $t\in J$ and $s\in J_t$,

$$2\varphi^{-1}(t)=\varphi^{-1}(t+s)+\varphi^{-1}(t-s),$$

hence, by the continuity of φ^{-1}, $\varphi^{-1}(u)=Au+B$ ($A\neq 0$, B are constants) if $u\in J$. Wherefrom, $\varphi\sim\chi_0$ and $\psi\sim\chi_0$ on I. Thus, for all nonvoid open intervals $K\subset I$, φ and ψ are differentiable, and their derivatives are nonzero constant functions.

In Case (ii) there exists $t_0\notin\sigma(\varphi^{-1})$, and with the notation $G:=\{t\in J\mid t\notin\sigma(\varphi^{-1})\}$, by Lemma 4.6, we have that G is a *nonvoid open* set. Thus, there exists a *nonvoid open interval* $\Delta\subset G\subset J$ such that, by Theorem 4.7, φ^{-1} is differentiable on Δ. Therefore, φ is differentiable on some nonvoid open interval $K_0\subset I$ and the corollary of Theorem 4.3 implies that $\varphi'(x)\neq 0$ if $x\in K_0$. Now consider the Matkowski–Sutô problem on the nonvoid open interval K_0. Then φ and ψ are interchangeable, and the same consideration gives that *there exists* a nonvoid open interval $K\subset K_0\subset I$ in which ψ is differentiable and $\psi'(x)\neq 0$ if $x\in K$. Thus, the existence of the desired subinterval is proven in this case as well. □

4.4. Continuous differentiability property of Matkowski–Sutô pairs

According to Theorem 4.8, it is sufficient to consider the case when a Matkowski–Sutô pair $(\varphi,\psi)\in\mathcal{CM}(I)^2$ has the following property: φ and ψ are differentiable on I, with nonvanishing first derivatives. Since φ' and ψ' have the Darboux property, we can assume that $\varphi'(x)>0$, $\psi'(x)>0$ for $x\in I$. Then, by Lemma 3.7, with the notations $J:=\varphi(I)$, $f:=\varphi'\circ\varphi^{-1}$, $g:=\psi'\circ\varphi^{-1}$, we have that the functions $f,g: J\to\mathbb{R}_+$ satisfy the following functional equation

$$2f\left(\frac{u+v}{2}\right)\big(g(v)-g(u)\big)=f(u)g(v)-f(v)g(u) \tag{74}$$

for every $u,v\in J$.

Definition 4.9. We will say that the function $h : J \to \mathbb{R}$ is an element of the set $\mathcal{D}(J)$ if

(i) $h = d \circ c$, where $c \in \mathcal{CM}(J)$, and, with the notation $I := c(J)$, $d : I \to \mathbb{R}$ is a *derivative function*, that is, there exists a differentiable function $D : I \to \mathbb{R}$ such that $D'(x) = d(x)$ holds for every $x \in I$;

(ii) $h(t) > 0$, if $t \in J$.

According to the above mentioned definition, the unknown functions f and g in equation (74) are elements of $\mathcal{D}(J)$.

We intend to prove the following statement.

Theorem 4.10. *If $f, g \in \mathcal{D}(J)$ satisfies the functional equation* (74) *for every $u, v \in J$, then there exists a nonvoid open interval $J_0 \subset J$ in which f is continuous.*

Proof. (i) If there exists a nonvoid open interval $J_0 \subset J$ such that f is constant on J_0, then the statement is true. If there exists a nonvoid open interval $J_0 \subset J$ such that g is constant on J_0, then let $g(t) =: k$ if $t \in J_0$, where $k > 0$ is a constant. Substitute arbitrary values $u, v \in J_0$ ($\subset J$) into (74). Then $f(u)k - f(v)k = 0$ for every $u, v \in J_0$, whence it follows that f is also a constant on J_0. Thus, f is continuous on J_0.

From now on we may assume that f and g are such functions that are *non-constant* on any nonvoid open subinterval $J_0 \subset J$. Denote by $\mathcal{D}_0(J)$ all those functions from $\mathcal{D}(J)$ for which it holds that they are non-constant on any nonvoid open interval $J_0 \subset J$. Hence, we only have to examine equation (74) for those functions f, g that belong to $\mathcal{D}_0(J)$.

(ii) Thus, let $f, g \in \mathcal{D}_0(J)$ satisfy (74) for every $u, v \in J$. Denote

$$C(g) := \big\{t \mid t \in J,\ g \text{ is continuous at } t\big\}.$$

Then g is of the form $d \circ c$, where d is a derivative and c is continuous and strictly monotone. Thus, g is continuous at the point $t \in J$ if d is continuous at $c(t)$. Since the derivative function d is of Baire class 0 or 1, thus, according to Baire's theorem ([44], [46], [36]), the set of all points at which d is continuous is a *dense* and G_δ set in the interval $c(J)$, whence, because c is continuous and strictly monotone, $C(g)$ is also *dense G_δ set* in J.

Now we will show that there exist points $u_0, v_0 \in C(g)$ such that

$$g(u_0) \neq g(v_0). \tag{75}$$

Contrary to our assumption, suppose that $g(t) = k$ for every $t \in C(g)$, where $k > 0$ is constant. Then, by (74), with the substitution $u, v \in C(g)$, we have

$$f(u)k - f(v)k = 0,$$

whence it follows that $f(t) = l$ for every $t \in C(g)$, where $l > 0$ is constant.

Because of the property of the set $C(g)$ for all $u \in J$, there exists $v \in C(g)$ such that $\frac{u+v}{2} \in C(g)$. Thus, by (74),

$$2l(k - g(u)) = f(u)k - lg(u)$$

for every $u \in J$. From this

$$f(u) = \frac{2lk - lg(u)}{k} \quad \text{if } u \in J. \tag{76}$$

If we substitute the function f of the form (76) back into the equation (74), then we have

$$2\,\frac{2lk - lg\left(\frac{u+v}{2}\right)}{k}\,(g(v) - g(u)) = \frac{2lk - lg(u)}{k}\,g(v) - \frac{2lk - lg(v)}{k}\,g(u)$$

for every $u, v \in J$, wherefrom, with an easy computation, we obtain

$$\left(k - g\left(\frac{u+v}{2}\right)\right)(g(v) - g(u)) = 0 \tag{77}$$

for every $u, v \in J$.

Now let $v_0 \in J$ be fixed in such way that $m := g(v_0) \neq k$ holds. Such a v_0 exists since g is non-constant.

On the other hand, for every $t \in J$ and for every $\varepsilon > 0$, for which $]t - \varepsilon, t + \varepsilon[\subset J$ holds, there exists $u \in]t - \varepsilon, t + \varepsilon[\subset J$ such that

$$g\left(\frac{u + v_0}{2}\right) \neq k.$$

This last statement is valid because g is non-constant on any proper subinterval. Thus, by (77), we have $g(u) = g(v_0) = m$. So, in any neighborhood of *any* point $t \in J$ there exists u such that $g(u) = m$, and there exists s such that $g(s) = k \neq m$, therefore, we deduce that g is a nowhere continuous function, which is a contradiction.

(iii) We proved in the previous part (ii) that if $f, g \in \mathcal{D}_0(J)$ are solutions of (74), then there exist $u_0, v_0 \in C(g)$ such that

$$g(u_0) \neq g(v_0).$$

Therefore, there exist neighborhood $U \subset J$ of u_0 and neighborhood $V \subset J$ of v_0 such that, for any $u \in U$ and $v \in V$, we have $g(u) \neq g(v)$, because g is continuous at $u_0 \neq v_0$. Thus, by (74), for any $u \in U$ and $v \in V$, we get

$$f\left(\frac{u+v}{2}\right) = \frac{1}{2}\,\frac{f(u)g(v) - f(v)g(u)}{g(v) - g(u)}. \tag{78}$$

From equation (78), due to the given condition, it follows that f is continuous on a nonvoid open interval. This is a consequence of JÁRAI's work [31], [32]. Járai's results are so general that it would be tiresome to check its specialization for a certain case, so we derive here a direct proof using Járai's ideas. With the notation

$$C(f) := \big\{u \mid u \in J,\ f \text{ is continuous at } u\big\}$$

let

$$U_1 := C(f) \cap C(g) \cap U \quad \text{and} \quad V_1 := C(f) \cap C(g) \cap V.$$

Then the sets U_1 and V_1 are disjoint and are of *second category* and have the *Baire's property* ([46]). Thus, according to Piccard's theorem ([51], [35], [57]–[59], [28], [22]), there exists a nonvoid bounded open interval J_0 such that $J_0 \subset \frac{U_1+V_1}{2} \subset J$. Whence, by (78), we have that f is *continuous* on J_0. Indeed, let $t \in J_0$ be arbitrary and $t_n \to t$ ($t_n \in J$). Then, we may suppose that $t_n \in J_0$. Thus, $t_n = \frac{u_n+v_n}{2}$, where $u_n \in U_1$ and $v_n \in V_1$, and $t = \frac{u+v}{2}$, where $u \in U_1$ and $v \in V_1$. The *bounded* sequences $\{u_n\}$ and $\{v_n\}$ have convergent subsequences $\{u_{n_k}\}$ and $\{v_{n_k}\}$, and because of $u_{n_k} \in U_1$ and $v_{n_k} \in V_1$, $\lim_{k\to\infty} u_{n_k} = u^* \in U$ and $\lim_{k\to\infty} v_{n_k} = v^* \in V$. Since $t_{n_k} = \frac{u_{n_k}+v_{n_k}}{2} \to \frac{u^*+v^*}{2}$ ($k \to \infty$) and $\frac{u^*+v^*}{2} = \frac{u+v}{2}$, therefore,

$$\begin{aligned}
\lim_{n\to\infty} f(t_n) &= \lim_{n\to\infty} f\left(\frac{u_n+v_n}{2}\right) = \lim_{k\to\infty} f\left(\frac{u_{n_k}+v_{n_k}}{2}\right)\\
&= \lim_{k\to\infty} \frac{1}{2}\,\frac{f(u_{n_k})g(v_{n_k}) - f(v_{n_k})g(u_{n_k})}{g(v_{n_k}) - g(u_{n_k})}\\
&= \frac{1}{2}\,\frac{f(u^*)g(v^*) - f(v^*)g(u^*)}{g(v^*) - g(u^*)} = f\left(\frac{u^*+v^*}{2}\right) = f\left(\frac{u+v}{2}\right) = f(t),
\end{aligned}$$

that is, f is continuous at t. □

Now we are ready to state the main regularity theorem on the solutions of the Matkowski–Sutô equation.

Theorem 4.11. *If $(\varphi,\psi)\in\mathcal{CM}(I)^2$ is a Matkowski–Sutô pair, then there exists a nonvoid open interval $K\subset I$ such that φ and ψ are continuously differentiable on K and their derivatives do not vanish in K.*

Proof. Let $(\varphi,\psi)\in\mathcal{CM}(I)^2$ be a Matkowski–Sutô pair. Then, because of Theorem 4.8, there exists a nonvoid open interval $K_1\subset I$ on which φ and ψ are differentiable and $\varphi'(x)\neq 0$, $\psi'(x)\neq 0$ if $x\in K_1$. We can assume that $\varphi'(x)>0$, $\psi'(x)>0$ if $x\in K_1$. Then, by Lemma 3.7 with the notations $J:=\varphi(K_1)$, $f:=\varphi'\circ\varphi^{-1}$, and $g:=\psi'\circ\varphi^{-1}$, we have that the functional equation (32) (and (74)) holds, where $f,g\in\mathcal{D}(K_1)$. Thus, according to Theorem 4.10, there exists a nonvoid open interval $J_0\subset J$ in which f is continuous. This means that the function $f:=\varphi'\circ\varphi^{-1}:J\to\mathbb{R}_+$ is *continuous* on $J_0\subset J$. Whence, $K_2:=\varphi^{-1}(J_0)\subset K_1\subset I$ is a nonvoid open interval, and

$$\varphi'(x)=f\circ\varphi(x)$$

for every $x\in K_2$. Hence, by the continuity of the composite function, φ' is *continuous* on the nonvoid open interval $K_2\subset I$.

Now consider the pair $(\varphi,\psi)\in\mathcal{CM}(K_2)^2$ which is obviously a Matkowski–Sutô pair on K_2. Then φ is continuously differentiable on K_2 and $\varphi'(x)>0$ if $x\in K_2$. Applying our result for ψ, we have that there exists a nonvoid open subinterval $K\subset K_2$ such that ψ is continuously differentiable on K, and $\psi'(x)>0$ if $x\in K$. Then the statement of the theorem holds in the set K, hence, the proof is completed. □

4.5. The solution of the Matkowski–Sutô problem

The main result of our investigations so far is the solution of the original Matkowski–Sutô problem.

Theorem 4.12. *If $(\varphi,\psi)\in\mathcal{CM}(I)^2$ is a Matkowski–Sutô pair, that is, the functions $\varphi,\psi\in\mathcal{CM}(I)$ satisfy the functional equation*

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)+\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)=x+y \tag{79}$$

for every $x, y \in I$, then there exists $p \in \mathbb{R}$ such that

$$\varphi \sim \chi_p, \qquad \psi \sim \chi_{-p}, \qquad \text{on} \quad I. \tag{80}$$

This means that (79) *is fulfilled if and only if there exists $p \in \mathbb{R}$ such that*

$$A_\varphi(x,y) = S_p(x,y) \qquad A_\psi(x,y) = S_{-p}(x,y) \tag{81}$$

for every $x, y \in I$, where

$$S_p(x,y) := \begin{cases} \dfrac{x+y}{2} & \text{if } p = 0 \\ \dfrac{1}{p}\,\log\left(\dfrac{e^{px}+e^{py}}{2}\right) & \text{if } p \neq 0 \end{cases} \qquad (x, y \in I). \tag{82}$$

Proof. Because of Theorem 4.11, there exists a nonvoid open interval $K \subset I$ in which φ and ψ are *continuously differentiable and φ', ψ' do not vanish on K*. Hence, by Theorem 3.6, there exists $p \in \mathbb{R}$ such that $\varphi \sim \chi_p$, $\psi \sim \chi_{-p}$ hold on K, where the function χ_p is defined in (31). According to Theorem 3.14 (the extension theorem), this yields (80). Clearly, (81) is a reformulation of this statement. □

An immediate but interesting consequence of our main result is the following extension theorem.

Corollary 4.13. *If $(\varphi, \psi) \in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair, that is, φ and ψ satisfy the functional equation* (79) *then there exists a Matkowski–Sutô pair $(\widetilde{\varphi}, \widetilde{\psi}) \in \mathcal{CM}(\mathbb{R})$ such that $\widetilde{\varphi}|_I = \varphi$, $\widetilde{\psi}|_K = \psi$.*

Proof. If (79) holds, then, for some $p \in \mathbb{R}$, we have (80). Hence, there exist $a, b, c, d \in \mathbb{R}$ with $ac \neq 0$ such that

$$\varphi(x) = a\chi_p(x) + b, \qquad \psi(x) = c\chi_{-p}(x) + d.$$

for all $x \in I$. Defining $\widetilde{\varphi}, \widetilde{\psi} : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{\varphi}(x) = a\chi_p(x) + b, \quad \widetilde{\psi}(x) = c\chi_{-p}(x) + d \qquad (x \in \mathbb{R}),$$

we get the desired extension. □

It is also interesting to note that the means S_p playing crucial role in the solution are exactly the *translation invariant* means among quasi-arithmetic means.

Now we are going to examine the solution of the general Matkowski–Sutô problem stated in Section 2.3 (and in 4.1) for the class of quasi-arithmetic means.

Theorem 4.14. *If $M_i : I^2 \to I$ $(i = 1, 2, 3)$ are quasi-arithmetic means on I, then the identity*

$$M_1 = M_2 \otimes M_3 \tag{83}$$

holds on I^2 if and only if there exist $f \in \mathcal{CM}(I)$ and $p \in \mathbb{R}$ such that

$$\begin{aligned} M_1(x,y) &= A_f(x,y), \\ M_2(x,y) &= A_{\chi_p \circ f}(x,y), \\ M_3(x,y) &= A_{\chi_{-p} \circ f}(x,y) \end{aligned} \tag{84}$$

hold for every $x, y \in I$.

Proof. Then there exist generating functions $f_1, f_2, f_3 \in \mathcal{CM}(I)$ such that the invariance equation

$$A_{f_1}(x,y) = A_{f_1}\big(A_{f_2}(x,y), A_{f_3}(x,y)\big) \tag{85}$$

holds for every $x, y \in I$. Thus, with the notations $u := f_1(x)$, $v := f_1(y)$, $u, v \in f_1(I) =: J$, $\varphi := f_2 \circ f_1^{-1}$, $\psi := f_3 \circ f_1^{-1}$, (85) holds if and only if $(\varphi, \psi) \in \mathcal{CM}(I)^2$ is a Matkowski–Sutô pair. Therefore, by Theorem 4.12, there exists $p \in \mathbb{R}$ such that $\varphi \sim \chi_p$ and $\psi \sim \chi_{-p}$ on J. Whence, with the notation $f := f_1$, $f \in \mathcal{CM}(I)$, and because of

$$\varphi = f_2 \circ f_1^{-1} = f_2 \circ f^{-1}, \qquad \psi = f_3 \circ f_1^{-1} = f_3 \circ f^{-1},$$

we have

$$f_2 = \varphi \circ f \sim \chi_p \circ f, \qquad f_3 = \psi \circ f \sim \chi_{-p} \circ f$$

on I for some $p \in \mathbb{R}$. Thus, the equations (84) are satisfied, where

$$\begin{aligned} A_{\chi_p \circ f}(x,y) &= f^{-1}\big(S_p\big(f(x), f(y)\big)\big) \\ &= \begin{cases} f^{-1}\left(\dfrac{f(x)+f(y)}{2}\right), & \text{if } p = 0 \\ f^{-1}\left(\dfrac{1}{p}\,\log\left(\dfrac{e^{pf(x)}+e^{pf(y)}}{2}\right)\right), & \text{if } p \neq 0 \end{cases} \end{aligned}$$

for every $x, y \in I$. □

We note that this theorem is a considerable generalization of Theorems 3.1, 3.2, 3.3, and 3.5.

5. Applications and further problems

5.1. Some applications

Fist we consider the problem examined by Daróczy ([13]) and Daróczy–Páles ([18]). Let $I \subset \mathbb{R}$ be a nonvoid open interval and let $M : I^2 \to I$ be a mean. We ask when the mean M on I is a *quasi-arithmetic* and a *conjugate-arithmetic* mean at the same time?

If M is quasi-arithmetic mean on I, then there exists $\psi \in \mathcal{CM}(I)$ such that

$$M(x,y) = \psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right) \qquad (x,y \in I). \tag{86}$$

If M is a conjugate-arithmetic mean on I, then there exists $\varphi \in \mathcal{CM}(I)$ such that

$$M(x,y) = \varphi^{-1}\left(\varphi(x)+\varphi(y)-\varphi\left(\frac{x+y}{2}\right)\right) \qquad (x,y \in I). \tag{87}$$

Thus, based on (86) and (87), our question is when the functional equation

$$\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right) = \varphi^{-1}\left(\varphi(x)+\varphi(y)-\varphi\left(\frac{x+y}{2}\right)\right) \tag{88}$$

holds for every $x, y \in I$, where $\varphi, \psi \in \mathcal{CM}(I)$ are unknown functions.

The next result completely solves this problem. In order to describe the result, we need the following notations:

If $I \subset \mathbb{R}$ is a nonvoid open interval, then let

$$P_+(I) := \big\{\lambda \in \mathbb{R} \mid I+\lambda \subset \mathbb{R}_+\big\} \quad \textit{and} \quad P_-(I) := \big\{\mu \in \mathbb{R} \mid -I+\mu \subset \mathbb{R}_+\big\}.$$

We note that for *bounded* $I =]a,b[$ $(a,b \in \mathbb{R},\ a<b)$ we have

$$P_+(I) := \big\{\lambda \in \mathbb{R} \mid \lambda > -a\big\} \quad \textit{and} \quad P_-(I) := \big\{\mu \in \mathbb{R} \mid \mu > b\big\}.$$

If I is not bounded, then, in case of $I = \mathbb{R}$ we have $P_+(\mathbb{R}) = P_-(\mathbb{R}) = \emptyset$; if $I =]-\infty, b[$ $(b \in \mathbb{R})$, then $P_+(I) = \emptyset$, $P_-(I) = \big\{\mu \in \mathbb{R} \mid \mu > b\big\}$;

and at last, if $I =]a, \infty[$ $(a \in \mathbb{R})$, then $P_+(I) = \{\lambda \in \mathbb{R} \mid \lambda > -a\}$ and $P_-(I) = \emptyset$. Denote by H the harmonic mean defined, for $x, y \in \mathbb{R}_+$, by

$$H(x,y) := \frac{2xy}{x+y}.$$

Theorem 5.1. *The mean $M : I^2 \to I$ is a quasi-arithmetic and a conjugate-arithmetic mean on I if and only if either*

$$M(x,y) = \frac{x+y}{2} \qquad (x,y \in I), \tag{89}$$

or

$$M(x,y) = H(x+\lambda, y+\lambda) - \lambda \qquad (x,y \in I) \tag{90}$$

for some $\lambda \in P_+(I)$, or

$$M(x,y) = -H(-x+\mu, -y+\mu) + \mu \qquad (x,y \in I) \tag{91}$$

for some $\mu \in P_-(I)$.

Proof. As it follows from the previous results, we have to give the solutions of the functional equation (88) disregarding the equivalence of the unknown functions $\varphi, \psi \in \mathcal{CM}(I)$. In (88), let $u = \varphi(x)$, $v = \varphi(y)$ $(u, v \in \varphi(I) =: J)$ be arbitrary and $f := \varphi^{-1}$, $g := \psi \circ \varphi^{-1}$ $(f, g \in \mathcal{CM}(J))$. Then, by (88), for every $u, v \in J$

$$A_f(u,v) + A_g(u,v) = u + v \tag{92}$$

holds for the functions $f, g \in \mathcal{CM}(J)$, that is, the Matkowski–Sutô equation is satisfied for the pair $(f,g) \in \mathcal{CM}(J)^2$. Therefore, by Theorem 4.12, there exists $p \in \mathbb{R}$ such that $f(u) \sim \chi_p(u)$ if $u \in J$, that is,

$$\varphi^{-1}(u) \sim \begin{cases} u & \text{if } p = 0 \\ e^{pu} & \text{if } p \neq 0 \end{cases} \qquad (u \in J). \tag{93}$$

Hence, in case of $p = 0$, we have $\varphi^{-1}(u) = au + b$ $(u \in J;\ a \neq 0,\ b$ constants). Thus

$$\varphi(x) = \frac{x-b}{a} \sim x, \quad \text{if } x \in I. \tag{94}$$

If $p \neq 0$ in (93), then

$$\varphi^{-1}(u) = a\, e^{pu} + b,$$

where $a \neq 0$ and b are constant values. So

$$x = \operatorname{sgn} a\, e^{\log|a| + p\varphi(x)} + b. \tag{95}$$

Now there are two possible cases: (i) $\operatorname{sgn} a = +1$ or (ii) $\operatorname{sgn} a = -1$. In case (i), by (95), $x - b > 0$, that is, $\lambda := -b \in P_+(I)$ and

$$p\varphi(x) + \log|a| = \log(x + \lambda) \qquad (x \in I),$$

whence

$$\varphi(x) = \frac{1}{p}\log(x+\lambda) - \frac{\log|a|}{p} \;\sim\; \log(x+\lambda), \quad \text{if } x \in I. \tag{96}$$

In case (ii) $-x + b > 0$, that is $\mu := b \in P_-(I)$ and by (95)

$$p\varphi(x) + \log|a| = \log(-x + \mu) \qquad (x \in I),$$

whence

$$\varphi(x) = \frac{1}{p}\log(-x+\mu) - \frac{\log|a|}{p} \;\sim\; \log(-x+\mu), \quad \text{if } x \in I. \tag{97}$$

Then, in case of generating functions of type (94), we have the mean (89). In case of generating functions of type (96) and (97), the solutions are the means (90) and (91). These means are in fact conjugate-arithmetic and quasi-arithmetic on I at the same time. Thus, the proof is complete. □

Now consider an economic application. We formulate the problem as it follows: The amount S is divided up between two parties (goals) in same way. Let us say these are x and y, that is, $x + y = S$. Now, because of different reasons, the decision makers have to change their judgment. They agreed on the following principles:

(i) The new amounts depend on the old ones, that is, the new X and Y are some *functions* of the amounts x and y and the dividable amount S remains the same, that is, $X + Y = S = x + y$ holds.

(ii) The new amounts X and Y are the quasi-arithmetic means of x and y (or equivalently to this they have the property $\mathcal{QA}$). That is, $X, Y \in \mathcal{QA}$ and

$$X + Y = x + y.$$

(iii) The value of S can be changed from time to time, the x and y are arbitrary positive amounts satisfying the condition $x+y=S$, hence, the first two conditions have to hold for all possible cases. That is, the quasi-arithmetic means X and Y satisfy the Matkowski–Sutô equation. Thus, there exists $p \in \mathbb{R}$ such that, with the notation

$$S_p(x,y) := \begin{cases} \dfrac{x+y}{2}, & \text{if } p=0 \\ \dfrac{1}{p}\,\log\left(\dfrac{e^{px}+e^{py}}{2}\right), & \text{if } p\neq 0 \end{cases}$$

we have

$$X(x,y) = S_p(x,y) \quad \text{and} \quad Y(x,y) = S_{-p}(x,y) \quad (x,y \in \mathbb{R}_+).$$

The result shows that after all the decision makers have to rule on a parameter $p \in \mathbb{R}$, and according to it, they can determine the new dividing amounts X and Y.

With the *choice* $p=0$, they have $X=Y$, that is, the full amount has to be divided equally between the two parties. For any other case ($p \neq 0$) this is not true. For example, according to

$$\lim_{p\to-\infty} S_p(x,y) = \min\{x,y\}, \qquad \lim_{p\to\infty} S_p(x,y) = \max\{x,y\},$$

if first they decide to choose the value $p=-\infty$, then the first group gets the minimum of x and y and the second one gets the maximum. With the choice $p=+\infty$, we have the opposite result, that is, the first group gets the maximum and the second one gets the minimum. That is, the decision makers can change the original partitioning by simply reversing the different amounts originally given to the two parties.

At last, with the choice $p<0$ the first group always gets smaller value than the second one and in the case of $p>0$ the opposite state holds.

The following remark shows the importance of the economic point of view. In fact the decision making committee does not have to know who was favored by the original partition. The next table illustrates the

possible cases:

	1. group	2. group	altogether
1. decision	x	y	$x+y$
2. decision choice of $p \in \mathbb{R}$			
$p=0$	$\frac{x+y}{2}$	$\frac{x+y}{2}$	$x+y$
$p=-\infty$	$\min\{x,y\}$	$\max\{x,y\}$	$x+y$
$p=+\infty$	$\max\{x,y\}$	$\min\{x,y\}$	$x+y$
$p<0$	$\frac{1}{p}\log\left(\frac{e^{px}+e^{py}}{2}\right) < -\frac{1}{p}\log\left(\frac{e^{-px}+e^{-py}}{2}\right)$		$x+y$
$p>0$	$\frac{1}{p}\log\left(\frac{e^{px}+e^{py}}{2}\right) > -\frac{1}{p}\log\left(\frac{e^{-px}+e^{-py}}{2}\right)$		$x+y$

At last, from these cases providing $p \in \mathbb{R}$, we have that the absolute value of the difference of the amounts obtained by the two parties is the following:

$$|X-Y| = \begin{cases} 0, & \text{if } p=0 \\ |x-y|, & \text{if } p=-\infty \text{ or } p=+\infty \\ \left|\frac{1}{p}\log\left(\frac{1}{2}\left(1+\cosh p(y-x)\right)\right)\right|, & \text{if } p\neq 0 \end{cases}$$

Very likely it would be interesting to give some kind of economic reasoning for this result.

As the third and last application, we solve the problem posed by Daróczy–Maksa ([15]). Assume that the quasi-arithmetic means M and N satisfy the equation $M+N=2A$ on the set I^2. Then the following statement holds:

(D-M) *If there exist $a,b \in I$, $a \neq b$ such that $M(a,b) = \frac{a+b}{2}$, then $M(x,y) = N(x,y) = \frac{x+y}{2}$ for every $x,y \in I$.*

According to Theorem 4.12, this statement is true, since in this case there exists $p \in \mathbb{R}$ such that $M(x,y) = S_p(x,y)$ for every $x,y \in I$. Thus,

if exist $a, b \in I$, $a \neq b$ and $M(a,b) = \frac{a+b}{2}$, then necessarily $p = 0$ and the statement holds.

On the other hand the statement (D-M) implies Theorem 4.12. Let us assume that the statement (D-M) holds. Then

$$D(x,y) := M(x,y) - N(x,y) \qquad (x,y \in I)$$

is a continuous and symmetric function on I^2, thus, on the following connected set

$$\Delta := \big\{(x,y) \in I^2 \mid x < y\big\}$$

it is, too. Because of the (D-M) property, in case of $D(a,b) = D(b,a) = 0$ $(a, b \in I,\ a \neq b)$, we have $D(x,y) = 0$ for every $x, y \in I$, $x \neq y$. If now $D(x,y) \neq 0$ for every $x, y \in I$, $x \neq y$, then $D(x,y) \neq 0$ for $(x,y) \in \Delta$. Since $D : \Delta \to \mathbb{R}$ is continuous and Δ is connected, therefore, $D(x,y)$ is sign-preserving on Δ, that is, e.g. $D(x,y) > 0$ if $(x,y) \in \Delta$. By the symmetry, then $D(x,y) > 0$ for every $x, y \in I$, $x \neq y$, that is, M and N are strictly comparable on I. The other case is similar. Therefore, by result of article [15], it follows that there exists $p \neq 0$ such that $M = S_p$ and $N = S_{-p}$. Of course, this does not give a new proof of the Matkowski–Sutô problem, because we could not prove the statement (D-M) independently of the solution of the (M-S) problem.

5.2. Further problems and results

In this paragraph we discuss such problems which, directly or indirectly, are inspired by the Matkowski–Sutô problem.

(A) *The equality of quasi-arithmetic means of order α.*

If $\alpha \geq -1$ and $\varphi \in \mathcal{CM}(I)$, then the quantity

$$A_\varphi^{(\alpha)}(x,y) := \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y) + \alpha\varphi\left(\frac{x+y}{2}\right)}{2+\alpha}\right) \qquad (x,y \in I) \tag{98}$$

is called quasi-arithmetic mean of order α on I (see [17]). It is natural to ask when two such means will be equal, that is, when the identity

$$A_\varphi^{(\alpha)}(x,y) = A_\psi^{(\beta)}(x,y) \qquad (x,y \in I) \tag{99}$$

holds, where $\alpha, \beta \geq -1$ and $\varphi, \psi \in \mathcal{CM}(I)$. So far, we can solve this in two cases: (i) $\alpha = \beta$; (ii) $\alpha = -1$, $\beta = 0$. In case (i), we proved that (99) holds

if and only if $\varphi(x) \sim \psi(x)$ for $x \in I$ (unpublished). Case (ii) is already discussed in Section 5.1, and solved with the help of the answer given for the Matkowski–Sutô problem. The remaining cases can be formulated as it follows.

If $\beta = 0$ in (99), then

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)+\alpha\varphi\left(\frac{x+y}{2}\right)}{2+\alpha}\right) = \psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right),$$

wherefrom, with the notations $u = \varphi(x)$, $v = \varphi(y)$ $(u, v \in \varphi(I) =: J)$ and $f = \psi \circ \varphi^{-1}$, $g := \varphi^{-1}$ $(f, g \in \mathcal{CM}(J))$, we have

$$(2+\alpha)A_f(u,v) - \alpha\, A_g(u,v) = u + v. \tag{100}$$

As we saw it before, the cases $\alpha = 0$ and $\beta = -1$ are known, so, the interesting possibilities are $\alpha > 0$ or $0 > \alpha > -1$. The next problem refers to both of these cases:

Let $0 < \lambda < 1$ be fixed. Characterize all those quasi-arithmetic means M_1, M_2, M_3 on I, for which

$$M_1 = \lambda M_2 + (1-\lambda)M_3 \tag{101}$$

holds. From (100), in case of $\alpha > 0$, with the choices $\lambda := \frac{1}{2+\alpha}$, $M_1 := A_f$, $M_2 := A$, $M_3 := A_g$ we have (101); and from (100), in case of $0 > \alpha > -1$, with the choices $\lambda := \frac{2+\alpha}{2}$ $(\lambda \neq \frac{1}{2})$, $M_1 := A$, $M_2 := A_f$, $M_3 := A_g$ we have (101).

The examination of cases $\alpha\beta \neq 0$ and $\alpha \neq \beta$ $(\alpha \geq -1,\ \beta \geq -1)$ is an open problem.

(B) The generalized Matkowski–Sutô problem is the functional equation

$$A_\chi^{(\gamma)} = A_\varphi^{(\alpha)} \otimes A_\psi^{(\beta)} \tag{102}$$

where $\alpha, \beta, \gamma \geq -1$ are unknown constants, $\varphi, \psi, \chi \in \mathcal{CM}(I)$ are unknown functions. If $\alpha = \beta = \gamma = 0$ and $\chi(x) \sim x$ if $x \in I$, then this is the original Matkowski–Sutô problem. In the case $\alpha = \beta = \gamma$ $(\alpha \geq -1)$ and $\chi(x) \sim x$ if $x \in I$ Daróczy and Páles [17] dealt with the problem and solved it assuming the continuous differentiability of one of the generating function

φ or ψ. In the general case the problem is open, and we think that further results can be obtained by refining the methods used in the solution of the Matkowski–Sutô problem.

(C) Matkowski ([42]) dealt with the solution of the equation

$$M_1 = M_2 \otimes M_3 \tag{103}$$

with respect to Beckenbach–Gini means, where

$$M_i(x,y) := \frac{f_i(x)x + f_i(y)y}{f_i(x) + f_i(y)} \qquad (x, y \in I)$$

$(i = 1, 2, 3)$ and $f_i : I \to \mathbb{R}_+$ $(i = 1, 2, 3)$ are continuous functions. The complete solution of this problem is unknown. Similarly, it seems to be very hard to find all the solutions of the equation (103) in the class of quasi-arithmetic means weighted with weight functions.

(D) As we have seen it before, we can characterize the solutions of the equation (103) with respect to the class of one-parameter power means. Therefore, it would be interesting to consider this equation in case of Gini means or in case of Stolarsky means. The examination of these problems seems to be promising since the strong tools of analysis (differentiability, analytic property) are applicable.

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