

## On the distribution of primitive roots modulo a prime

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**Abstract.** Let  $p \geq 3$  be a prime. For each primitive root  $x$  modulo  $p$  with  $1 \leq x \leq p-1$ , it is clear that there exists one and only one primitive root  $\bar{x}$  modulo  $p$  with  $1 \leq \bar{x} \leq p-1$  such that  $x\bar{x} \equiv 1 \pmod{p}$ . Let  $\delta$  be a fixed positive number with  $0 \leq \delta \leq 1$ ,  $\mathcal{A}$  denotes the set of all primitive roots modulo  $p$  in interval  $[1, p]$ . For any fixed positive integers  $k$  and  $l$ , the main purpose of this paper is to study the asymptotic properties of the mean value

$$N(p, k, l, m, \delta) = \sum_{\substack{a \in \mathcal{A} \\ \left| \left\{ \frac{a^k}{p} \right\} - \left\{ \frac{\bar{a}^l}{p} \right\} \right| < \delta}} \left| \left\{ \frac{a^k}{p} \right\} - \left\{ \frac{\bar{a}^l}{p} \right\} \right|^m,$$

where  $m$  be any fixed non-negative real number,  $\{x\}$  denotes the fractional part of  $x$ , and give an interesting asymptotic formula.

### 1. Introduction

Let  $p \geq 3$  be a prime. For each primitive root  $x$  modulo  $p$  with  $1 \leq x \leq p-1$ , it is clear that there exists one and only one primitive root  $\bar{x}$  modulo  $p$  with  $1 \leq \bar{x} \leq p-1$  such that  $x\bar{x} \equiv 1 \pmod{p}$ . Let  $\delta$  be a fixed positive number with  $0 \leq \delta \leq 1$ ,  $\mathcal{A} = \mathcal{A}(p)$  denotes the set of all primitive roots modulo  $p$  in interval  $[1, p]$ . For any fixed positive integers  $k$  and  $l$ ,

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we define  $N(p, k, l, m, \delta)$  as follows:

$$(1) \quad N(p, k, l, m, \delta) = \sum_{\substack{a \in \mathcal{A} \\ \left| \left\{ \frac{a^k}{p} \right\} - \left\{ \frac{\bar{a}^l}{p} \right\} \right| < \delta}} \left| \left\{ \frac{a^k}{p} \right\} - \left\{ \frac{\bar{a}^l}{p} \right\} \right|^m$$

where  $m$  be any fixed non-negative real number,  $\{x\} = x - [x]$  denotes the fractional part of  $x$  ( $[x]$  denoting the integral part of  $x$ ). The main purpose of this paper is to study the asymptotic properties of  $N(p, k, l, m, \delta)$ .

About this problem, the second author [3] considered the case  $k = l = 1$ , and obtained a sharp asymptotic formula, which reads

$$\begin{aligned} N(p, m, \delta) &= \sum_{\substack{a \in \mathcal{A} \\ |a - \bar{a}| < \delta p}} |a - \bar{a}|^m \\ &= 2\phi(p-1)p^m \left( \frac{\delta^{m+1}}{m+1} - \frac{\delta^{m+2}}{m+2} \right) + O\left(p^{m+\frac{1}{2}+\epsilon}\right), \end{aligned}$$

where  $\phi(n)$  is the Euler function and  $\epsilon$  is any fixed positive number.

It is quite natural and interesting to consider the case of (1). In this paper, we use a trigonometric estimate and the G. I. Perel'muter's deep result to prove a sharp asymptotic formula for  $N(p, k, l, m, \delta)$  in the same setting as in paper [3].

Our main result is the following:

**Theorem.** *Let  $p \geq 3$  be a prime,  $\delta$  be a fixed positive number with  $0 \leq \delta \leq 1$  and  $m$  be any fixed non-negative real number. Then for any fixed positive integers  $k$  and  $l$ , we have the asymptotic formula*

$$N(p, k, l, m, \delta) = 2\phi(p-1) \left( \frac{\delta^{m+1}}{m+1} - \frac{\delta^{m+2}}{m+2} \right) + O\left(p^{\frac{1}{2}+\epsilon}\right),$$

where  $\phi(n)$  is the Euler function and  $\epsilon$  is any fixed positive number.

For  $m = 0$ , from this theorem we may immediately deduce the following:

**Corollary.** *For any prime  $p > 2$  and any fixed positive integer  $k$  and  $l$ , we have the asymptotic formula*

$$N(p, k, l, \delta) = \delta \cdot (2 - \delta) \cdot \phi(p-1) + O\left(p^{\frac{1}{2}+\epsilon}\right).$$

## 2. Some lemmas

To complete the proof of the theorem, we need following several lemmas.

**Lemma 1.** *Let  $p \geq 3$  be a prime,  $m$  and  $n$  be any fixed integers with  $(mn, p) = 1$ . Let  $\chi$  denotes a Dirichlet character modulo  $p$ . Then for any fixed positive integers  $k$  and  $l$ , we have the estimate*

$$\sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k + n\bar{a}^l}{p}\right) \ll \sqrt{p}.$$

PROOF. Taking rational functions  $R_1(a) = a$  and  $R_2(a) = \frac{m \cdot a^{k+l} + n}{a^l}$ . By Theorem 4 of [1] we may immediately obtain the estimate

$$\sum_{a=1}^{p-1} \chi(R_1(a)) e\left(\frac{R_2(a)}{p}\right) \ll \sqrt{p}.$$

This proves Lemma 1. □

**Lemma 2.** *Let modulo  $n \geq 3$  exists a primitive root. Then for each integer  $m$  with  $(m, n) = 1$ , we have the identity*

$$\begin{aligned} \sum_{k|\phi(n)} \frac{\mu(k)}{\phi} (k) \sum_{\substack{a=1 \\ (a,k)=1}}^k e\left(\frac{a \operatorname{ind} m}{k}\right) \\ = \begin{cases} \frac{\phi(n)}{\phi(\phi(n))}, & \text{if } m \text{ is a primitive root of } n; \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where  $\mu(n)$  be the Möbius function, and  $\operatorname{ind} m$  denotes the index of  $m$  relative to some fixed primitive root of  $n$ .

PROOF. (See Proposition 2.2 of reference [2].) □

**Lemma 3.** *Let  $p \geq 3$  be a prime,  $r$  and  $s$  be integers. Then for any fixed positive integers  $k$  and  $l$ , we have the estimate*

$$\sum_{a \in \mathcal{A}} e\left(\frac{r \cdot a^k + s \cdot \bar{a}^l}{p}\right) = O\left(p^{\frac{1}{2} + \epsilon}(r, s, p)^{\frac{1}{2}}\right).$$

PROOF. If  $p \mid s$  and  $p \mid r$ , then Lemma 3 is trivial. If  $p \mid r$  or  $p \mid s$ , but  $p \nmid r + s$ , then by the Gauss sum we also have the estimate of Lemma 3. So without loss of generality, we can assume  $(rs, p) = 1$ . Then from Lemma 1 and Lemma 2 we can easily deduce that

$$\begin{aligned}
\sum_{a \in \mathcal{A}} e\left(\frac{r \cdot a^k + s \cdot \bar{a}^l}{p}\right) &= \frac{\phi^2(p-1)}{(p-1)^2} \sum_{j|p-1} \sum_{h|p-1} \frac{\mu(j)\mu(h)}{\phi(j)\phi(h)} \\
&\quad \times \sum_{x=1}^j{}' \sum_{y=1}^h{}' \sum_{a=1}^{p-1} e\left(\frac{x \operatorname{ind} a}{j} + \frac{y \operatorname{ind} \bar{a}}{h}\right) e\left(\frac{r \cdot a^k + s \cdot \bar{a}^l}{p}\right) \\
&= \frac{\phi^2(p-1)}{(p-1)^2} \sum_{j|p-1} \sum_{h|p-1} \frac{\mu(j)\mu(h)}{\phi(j)\phi(h)} \\
&\quad \times \sum_{x=1}^j{}' \sum_{y=1}^h{}' \sum_{a=1}^{p-1} \chi(a; x, j) \chi(\bar{a}; y, h) e\left(\frac{r \cdot a^k + s \cdot \bar{a}^l}{p}\right) \\
&= \frac{\phi^2(p-1)}{(p-1)^2} \sum_{j|p-1} \sum_{h|p-1} \frac{\mu(j)\mu(h)}{\phi(j)\phi(h)} \\
&\quad \times \sum_{x=1}^j{}' \sum_{y=1}^h{}' \sum_{a=1}^{p-1} \chi(a; x, j) \overline{\chi(a; y, h)} e\left(\frac{r \cdot a^k + s \cdot \bar{a}^l}{p}\right) \\
&\ll \frac{\phi^2(p-1)}{(p-1)^2} \sum_{j|p-1} \sum_{h|p-1} |\mu(j)| \cdot |\mu(h)| p^{\frac{1}{2}} \\
&\ll \frac{\phi^2(p-1)}{(p-1)^2} \cdot 4^{\omega(p-1)} \cdot p^{\frac{1}{2}} \ll p^{\frac{1}{2}+\epsilon},
\end{aligned}$$

where  $\chi(a; x, j) = e\left(\frac{x \operatorname{ind} a}{j}\right)$  denotes a Dirichlet character modulo  $p$ ,  $\omega(n)$  denotes the number of all different prime divisors of  $n$ ,  $\epsilon$  is any fixed positive number,  $\sum_{x=1}^j{}'$  denotes the summation over all  $1 \leq x \leq j$  with  $(x, j) = 1$ .

This proves Lemma 3. □

**Lemma 4.** *Let  $p \geq 3$  be a prime,  $m$  be any fixed non-negative real number. Then for any fixed real number  $0 \leq \delta \leq 1$ , we have the estimate*

$$\sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \left| \sum_{\substack{c=1 \\ |c-d| < \delta p}}^{p-1} \sum_{d=1}^{p-1} |c-d|^m e\left(\frac{-rc-sd}{p}\right) \right| = O(p^{2+m} \ln^2 p).$$

PROOF. First note the trigonometric identity

$$(2) \quad \sum_{a=1}^n e(ax) = e\left(\frac{(n+1)x}{2}\right) \frac{\sin \pi n x}{\sin \pi x}.$$

So from (2) we obtain

$$\begin{aligned} (3) \quad & \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \left| \sum_{\substack{c=1 \\ |c-d| < \delta p}}^{p-1} \sum_{d=1}^{p-1} |c-d|^m e\left(\frac{-rc-sd}{p}\right) \right| \\ & \ll \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \left| \sum_{w=0}^{[\delta p]} w^m \sum_{\substack{c=1 \\ c-d=w}}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{-rc-sd}{p}\right) \right| \\ & \ll \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \left| \sum_{w=0}^{[\delta p]} w^m \sum_{d=1}^{p-1-w} e\left(\frac{-r(d+w)-sd}{p}\right) \right| \\ & \ll \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \left| \sum_{w=0}^{[\delta p]} w^m \cdot e\left(\frac{-rw}{p}\right) \sum_{d=1}^{p-1-w} e\left(\frac{-(r+s)d}{p}\right) \right| \\ & \ll \sum_{r=1}^{p-1} \left| \sum_{w=0}^{[\delta p]} w^m \cdot e\left(\frac{-rw}{p}\right) \cdot (p-1-w) \right| \\ & \quad + \sum_{\substack{r=1 \\ r+s \neq p}}^{p-1} \sum_{s=1}^{p-1} \left| \sum_{w=0}^{[\delta p]} w^m \cdot e\left(\frac{-rw}{p}\right) \cdot e\left(\frac{-(r+s)}{p}\right) \right. \\ & \quad \left. \cdot \frac{e\left(\frac{-(r+s)(p-1-w)}{p}\right) - 1}{e\left(\frac{-(r+s)}{p}\right) - 1} \right| \end{aligned}$$

$$\begin{aligned} &\ll \sum_{r=1}^{p-1} \left| \sum_{w=0}^{[\delta p]} w^m e\left(\frac{-rw}{p}\right) \cdot (p-1-w) \right| + \sum_{\substack{r=1 \\ r+s \neq p}}^{p-1} \sum_{s=1}^{p-1} \frac{1}{\left| e\left(\frac{-(r+s)}{p}\right) - 1 \right|} \\ &\times \left| \sum_{w=0}^{[\delta p]} w^m e\left(\frac{-rw - (r+s)(p-1-w)}{p}\right) - \sum_{w=0}^{[\delta p]} w^m \cdot e\left(\frac{-rw}{p}\right) \right|. \end{aligned}$$

Noting that the trigonometric estimate

$$(4) \quad \sum_{m \leq M} m^k e(mx) \leq M^k \cdot \min\left(M, \frac{1}{|\sin \pi x|}\right), \quad \text{if } k \geq 0.$$

From (3) and (4) we immediately get

$$\begin{aligned} &\sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \left| \sum_{\substack{c=1 \\ |c-d| < \delta p}}^{p-1} \sum_{d=1}^{p-1} |c-d|^m e\left(\frac{-rc-sd}{p}\right) \right| \\ &\ll \sum_{r=1}^{p-1} \frac{p^{m+1}}{|\sin \frac{\pi r}{p}|} + \sum_{\substack{r=1 \\ r+s \neq p}}^{p-1} \sum_{s=1}^{p-1} \frac{1}{|\sin \frac{\pi(r+s)}{p}|} \left[ \frac{p^m}{|\sin \frac{\pi r}{p}|} + \frac{p^m}{|\sin \frac{\pi s}{p}|} \right] \\ &\ll p^{2+m} \ln p + p^m \cdot \sum_{r=1}^{p-1} \frac{1}{|\sin \frac{\pi r}{p}|} \sum_{\substack{s=1 \\ s \neq p-r}}^{p-1} \frac{1}{|\sin \frac{\pi(r+s)}{p}|} \\ &\ll p^{2+m} \ln^2 p. \end{aligned}$$

This proves Lemma 4. □

### 3. Proof of the theorem

In this section, we complete the proof of the Theorem. First note the trigonometric identity

$$\sum_{r=1}^q e\left(\frac{rn}{q}\right) = \begin{cases} q, & \text{if } q \mid n; \\ 0, & \text{if } q \nmid n. \end{cases}$$

and the identity

$$\begin{aligned} \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \left( \sum_{a \in \mathcal{A}} e \left( \frac{r \cdot p \left\{ \frac{a^k}{p} \right\} + s \cdot p \left\{ \frac{\bar{a}^l}{p} \right\}}{p} \right) \right) \\ = \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \left( \sum_{a \in \mathcal{A}} e \left( \frac{r \cdot a^k + s \cdot \bar{a}^l}{p} \right) \right). \end{aligned}$$

From these trigonometric identities, Lemma 3 and Lemma 4, we have

$$\begin{aligned} N(p, k, l, m, \delta) &= \sum_{\substack{a \in \mathcal{A} \\ \left| \left\{ \frac{a^k}{p} \right\} - \left\{ \frac{\bar{a}^l}{p} \right\} \right| < \delta}} \left| \left\{ \frac{a^k}{p} \right\} - \left\{ \frac{\bar{a}^l}{p} \right\} \right|^m \\ &= \frac{1}{p^2} \sum_{r,s=1}^p \sum_{a \in \mathcal{A}} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \frac{1}{p^m} |c-d|^m e \left( \frac{r \left( p \left\{ \frac{a^k}{p} \right\} - c \right)}{p} \right) e \left( \frac{s \left( p \left\{ \frac{\bar{a}^l}{p} \right\} - d \right)}{p} \right) \\ &= \frac{1}{p^{2+m}} \sum_{r,s=1}^p \left( \sum_{a \in \mathcal{A}} e \left( \frac{r \cdot p \left\{ \frac{a^k}{p} \right\} + s \cdot p \left\{ \frac{\bar{a}^l}{p} \right\}}{p} \right) \right) \\ &\quad \times \sum_{\substack{c=1 \\ |c-d| < \delta p}}^{p-1} \sum_{d=1}^{p-1} |c-d|^m e \left( \frac{-rc - sd}{p} \right) \\ &= \frac{1}{p^{2+m}} \sum_{r,s=1}^p \left( \sum_{a \in \mathcal{A}} e \left( \frac{r \cdot a^k + s \cdot \bar{a}^l}{p} \right) \right) \sum_{\substack{c=1 \\ |c-d| < \delta p}}^{p-1} \sum_{d=1}^{p-1} |c-d|^m e \left( \frac{-rc - sd}{p} \right) \\ &= \frac{1}{p^{2+m}} \sum_{a \in \mathcal{A}} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} |c-d|^m \\ &\quad + \frac{1}{p^{2+m}} \sum_{r=1}^{p-1} \left( \sum_{a \in \mathcal{A}} e \left( \frac{r \cdot a^k}{p} \right) \right) \cdot \sum_{\substack{c=1 \\ |c-d| < \delta p}}^{p-1} \sum_{d=1}^{p-1} |c-d|^m e \left( \frac{-rc}{p} \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{p^{2+m}} \sum_{s=1}^{p-1} \left( \sum_{a \in \mathcal{A}} e \left( \frac{s \cdot \bar{a}^l}{p} \right) \right) \cdot \sum_{\substack{c=1 \\ |c-d| < \delta p}}^{p-1} \sum_{d=1}^{p-1} |c-d|^m e \left( \frac{-sd}{p} \right) \\
& + \frac{1}{p^{2+m}} \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \left( \sum_{a \in \mathcal{A}} e \left( \frac{r \cdot a^k + s \cdot \bar{a}^l}{p} \right) \right) \\
& \times \sum_{\substack{c=1 \\ |c-d| < \delta p}}^{p-1} \sum_{d=1}^{p-1} |c-d|^m e \left( \frac{-rc - sd}{p} \right) \\
& = \frac{1}{p^{2+m}} \cdot \phi(p-1) \left( 2 \cdot \sum_{w=0}^{[\delta p]} \sum_{\substack{c=1 \\ c-d=w}}^{p-1} \sum_{d=1}^{p-1} w^m \right) + O(1) \\
& + O \left( p^{-2-m+\frac{1}{2}+\epsilon} \cdot \sum_{r=1}^{p-1} \left| \sum_{\substack{c=1 \\ |c-d| < \delta p}}^{p-1} \sum_{d=1}^{p-1} |c-d|^m e \left( \frac{-rc}{p} \right) \right| \right) \\
& + O \left( p^{-2-m+\frac{1}{2}+\epsilon} \cdot \sum_{r=1}^{p-1} \sum_{s=1}^{p-1} \left| \sum_{\substack{c=1 \\ |c-d| < \delta p}}^{p-1} \sum_{d=1}^{p-1} |c-d|^m e \left( \frac{-rc - sd}{p} \right) \right| \right) \\
& = \frac{1}{p^{2+m}} \cdot \phi(p-1) \left( 2 \cdot \sum_{w=0}^{[\delta p]} w^m \cdot (p-1-w) \right) + O(1) \\
& + O \left( p^{-2+\frac{1}{2}+\epsilon} \cdot \sum_{c=1}^{p-1} (\delta p + c) \cdot \frac{1}{\left| \sin \frac{\pi c}{p} \right|} \right) + O \left( p^{\frac{1}{2}+\epsilon} \right) \\
& = \frac{2}{p^{2+m}} \cdot \phi(p-1) \left( \frac{\delta^{m+1} p^{m+2}}{m+1} - \frac{\delta^{m+2} p^{m+2}}{m+2} + O(p^{m+1}) \right) + O \left( p^{\frac{1}{2}+\epsilon} \right) \\
& = 2 \cdot \phi(p-1) \left( \frac{\delta^{m+1}}{m+1} - \frac{\delta^{m+2}}{m+2} \right) + O \left( p^{\frac{1}{2}+\epsilon} \right).
\end{aligned}$$

This completes the proof of Theorem.

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