Publ. Math. Debrecen
61 / 3-4 (2002), 439–453

Common fixed point theorems for single-valued and multi-valued mappings

By ZEQING LIU (Liaoning), YUGUANG XU (Yunnan) and YEOL JE CHO (Chinju)

Abstract. In this paper, we prove some common fixed point theorems for single-valued and multi-valued mappings which extend, improve and unify a multitude of the corresponding results by FISHER [1]–[10], FISHER and SESSA [11], JUNGCK [12], KIM, KIM, LEEM and UME [14], LIU [15], OHTA and NIKAIDO [16] and others. At the same time, we correct errors for the results in [14], [16] and [18].

1. Introduction

Let (X,d) be a metric space and f, g be selfmappings of X. Let W and N denote the sets of nonnegative integers and positive integers, respectively. For $x,y \in X$ and $A,B \subset X$, we define some notations as follows:

$$O_f(x) = \{f^n x : n \in W\}, \quad O_f(x,y) = O_f(x) \cup O_f(y),$$

$$O_{f,g}(x) = \{f^n g^m x : n,\ m \in W\}, \quad O_{f,g}(x,y) = O_{f,g}(x) \cup O_{f,g}(y),$$

$$D(A,B) = \inf\{d(a,b) : a \in A,\ b \in B\},$$

$$\delta(A,B) = \sup\{d(a,b) : a \in A,\ b \in B\}, \quad \delta(A,A) = \delta(A),$$

$$H(A,B) = \max\big\{\sup\{D(a,B) : a \in A\}, \quad \sup\{D(A,b) : b \in B\}\big\},$$

Mathematics Subject Classification: 47H10, 54H25.
Key words and phrases: common fixed point, closed mapping, complete metric space, compact metric space.
The third author wishes to acknowledge the financial support of the Korea Research Foundation (KRF-2000-DP0013).

$$
\begin{aligned}
CB(X) &= \{A : A \text{ is a nonempty bounded closed subset of } X\},\\
CL(X) &= \{A : A \text{ is a nonempty closed subset of } X\},\\
B(X) &= \{A : A \text{ is a nonempty bounded subset of } X\},\\
C_f &= \{h : h : X \to X \text{ is a mapping satisfying } hf = fh\},\\
H_f &= \Big\{h : h : X \to X \text{ is a mapping satisfying}\\
&\qquad h\Big(\bigcap_{n\in N} f^n(X)\Big) \subset \bigcap_{n\in N} f^n(X)\Big\}.
\end{aligned}
$$

The mapping f is called a closed mapping if $y = fx$ whenever $\{x_n\}_{n\in N} \subset X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} fx_n = y$ for some $x, y \in X$. For each $t \in [0, +\infty)$, $[t]$ denotes the largest integers not exceeding t. Let

$$
\begin{aligned}
\Phi = \{\phi : \phi : [0, +\infty) \to [0, +\infty) &\text{ is upper semicontinuous}\\
\text{and nondecreasing and } \phi(t) < t &\text{ for } t > 0\}.
\end{aligned}
$$

A number of generalizations of the well-known Banach contraction principle have received much attention in recent years. For instance, see [1]–[22]. Kim, Kim, Leem and Ume [14] considered the following conditions:
(1.1) there exists $m, n \in N$ and $r \in [0, 1)$ such that, for every $x, y \in X$,

$$d((fg)^m x, (fg)^n y) \le r\delta(O_{f,g}(x, y)),$$

(1.2) there exists $m, n \in W$ such that, for any distinct $x, y \in X$,

$$d((fg)^m x, (fg)^n y) < \delta(O_{f,g}(x, y)),$$

and established two common fixed point theorems. Rehman and Ahmad [18] extended the principle to multivalued mappings.

In this paper, we consider the following more general conditions (1.3) and (1.4) instead of (1.1) and (1.2), respectively:
(1.3) there exists $m, n, p, q \in N$ and $\phi \in \Phi$ such that, for any $x, y \in X$,

$$d(f^m g^n x, f^p g^q y) \le \phi(\delta(O_{f,g}(x, y))),$$

(1.4) there exists $m, n, p, q \in W$ with $m + p, n + q \in N$ such that, for any $x, y \in X$ with $f^m g^n x \ne f^p g^q y$,

$$d(f^m g^n x, f^p g^q y) < \delta\Big(\bigcup_{h\in H_{fg}} h(O_{f,g}(x, y))\Big),$$

and obtain common fixed point theorems. On the other hand, we point out that Theorem 2.4 of [14] is false and all the results of [18] are meaningless.

Lemma 1.1 [20]. *Let $\phi \in \Phi$. Then, for every $t > 0$, $\phi(t) < t$ if and only if $\lim_{n\to\infty} \phi^n(t) = 0$, where ϕ^n denotes the composition of ϕ with itself n times.*

Lemma 1.2. *Let f be a closed mapping from a compact metric space (X, d) into itself and $A = \bigcap_{n\in N} f^n(X)$. Then*

(i) $\{f^n : n \in W\} \subset C_f \subset H_f$;

(ii) *A is a nonempty compact subset of X;*

(iii) $A = f(A)$;

(iv) $\delta(f^n X) \downarrow \delta(A)$ *as* $n \to \infty$.

Proof. Let g be in C_f. Then

$$g(A) = g\Big(\bigcap_{n\in N} f^n(X)\Big) \subset \bigcap_{n\in N} gf^n(X) \subset \bigcap_{n\in N} f^n(X) = A,$$

which implies that $g \in H_f$ and so $C_f \subset H_f$. Obviously $\{f^n : n \in W\} \subset C_f$.

Assume that $\{x_n\}_{n\in N}$ is a sequence in X with $\lim_{n\to\infty} fx_n = a \in X$. The compactness of X ensures that there exists a subsequence $\{x_{n_k}\}_{k\in N}$ such that $\lim_{k\to\infty} x_{n_k} = t \in X$ and so the closedness of f implies that $a = ft$. Thus $f(X)$ is a closed subset of X. Since X is compact, so is $f(X)$. Similarly, we infer that $f^n(X)$ is compact for any $n \geq 2$. It is easy to see that A is a nonempty compact subset of X. It follows from (i) that $f(A) \subset A$. Conversely, for any $a \in A$ and $n \in N$, there exists $a_n \in f^{n-1}(X)$ with $fa_n = a$. From the compactness of X, we may (by selecting a subsequence, if necessary) assume that $\lim_{n\to\infty} a_n = t \in X$. In view of $\{a_k\}_{k\geq n+1} \subset f^n(X)$ and the compactness of $f^n(X)$, we immediately conclude that $t \in f^n(X)$ for all $n \in N$. That is, $t \in A$. Since $ft = a$, then $A \subset f(A)$. Therefore, we have $A = f(A)$. Since $\{\delta(f^n(X))\}_{n\in N}$ is nonincreasing and bounded in below, $\{\delta(f^n(X))\}_{n\in N}$ is convergent. By the compactness of $f^n(X)$, there exist $x_n, y_n \in f^n(X)$ such that $d(x_n, y_n) = \delta(f^n(X))$. Of course, we may extract subsequences $\{x_{n_k}\}_{k\in N}$, $\{y_{n_k}\}_{k\in N}$ of $\{x_n\}$, $\{y_n\}$ such that $x_{n_k} \to x$, $y_{n_k} \to y$ as $k \to \infty$, respectively. Note that $x_{n_i}, y_{n_i} \in f^{n_k}(X)$ and that $f^{n_k}(X)$ is closed for

$i \geq k \geq 1$. This implies that $x, y \in f^{n_k}(X)$ for $k \in N$. Consequently, $x, y \in \bigcap_{k \in N} f^{n_k}(X) = A$. It follows that

$$\delta(A) \leq \lim_{k\to\infty} \delta(f^{n_k}(X)) = \lim_{k\to\infty} d(x_{n_k}, y_{n_k}) = d(x,y) \leq \delta(A),$$

which implies that

$$\delta(A) = \lim_{k\to\infty} \delta(f^{n_k}(X)) = \lim_{n\to\infty} \delta(f^n(X)).$$

This completes the proof. □

2. Common fixed point theorems for single-valued mappings

Now, we give some common fixed point theorems for commuting single-valued mappings.

Theorem 2.1. *Let f, g be commuting mappings from a complete metric space (X,d) into itself and fg be closed. Assume that $O_{f,g}(x)$ is bounded for all $x \in X$ and (1.3) holds. Then f and g have a unique common fixed point $w \in X$ and $\lim_{i\to\infty}(fg)^i f^a g^b x = w$ for all $x \in X$ and $a, b \in \{0,1\}$. Moreover,*

$$\max\{d((fg)^i f^a g^b x, w) : a, b \in \{0,1\}\} \leq \phi^{[\frac{i}{k}]}(\delta(O_{f,g}(x)))$$

for all $i \in N$, where $k = \max\{m,n,p,q\}$.

Proof. For any $i, j, s, t, h \in W$, it follows from (1.3) that

$$\begin{aligned} &d(f^{i+k+s}g^{i+k+t}x, f^{i+k+j}g^{i+k+h}x) \\ &\quad \leq \phi(\delta(O_{f,g}(f^{i+k-m+s}g^{i+k-n+t}x, f^{i+k-p+j}g^{i+k-q+h}x))) \\ &\quad \leq \phi(\delta(O_{f,g}(f^{i+s}g^{i+t}x, f^{i+j}g^{i+h}x))) \\ &\quad \leq \phi(\delta(O_{f,g}((fg)^i x))), \end{aligned}$$

which implies that

$$\delta(O_{f,g}((fg)^{i+k}x)) \leq \phi(\delta(O_{f,g}((fg)^i x))) \tag{2.1}$$

for all $i \in W$. We now write $i = sk + t$ for some $s, t \in W$ with $t \leq k-1$. (2.1) ensures that

$$\begin{aligned}
\delta(O_{f,g}((fg)^i x)) &\leq \phi(\delta(O_{f,g}((fg)^{(s-1)k+t}x))) \\
&\leq \phi^2(\delta(O_{f,g}((fg)^{(s-2)k+t}x))) \\
&\leq \cdots \\
&\leq \phi^s(\delta(O_{f,g}((fg)^t x))) \\
&\leq \phi^s(\delta(O_{f,g}(x))).
\end{aligned} \tag{2.2}$$

It follows from Lemma 1.1 and the boundedness of $O_{f,g}(x)$ and (2.2) that

$$\lim_{i\to\infty} \delta(O_{f,g}((fg)^i x))) = 0, \tag{2.3}$$

which means that $\{(fg)^i x\}_{i\in N}$ is a Cauchy sequence in X. By completeness of X, there exists $w \in X$ such that $\lim_{i\to\infty}(fg)^i x = w$. Note that

$$\begin{aligned}
d((fg)^i f^a g^b x, w) &\leq d((fg)^i f^a g^b x, (fg)^i x) + d((fg)^i x, w) \\
&\leq \delta(O_{f,g}((fg)^i x)) + d((fg)^i x, w)
\end{aligned}$$

for $a, b \in \{0, 1\}$. By (2.3) we have $\lim\limits_{i\to\infty} d((fg)^i f^a g^b x, w) = 0$ for $a, b \in \{0, 1\}$. This implies that

$$w = \lim_{i\to\infty} (fg)^i x = \lim_{i\to\infty} fg(fg)^i x. \tag{2.4}$$

Since fg is closed, we have $w = fgw$. For any $i, j, s, t \in W$, by (1.3), we have

$$\begin{aligned}
d(f^i g^j w, f^s g^t w) &= d(f^{i+k} g^{j+k} w, f^{s+k} g^{t+k} w) \\
&\leq \phi(\delta(O_{f,g}(f^i g^j w, f^s g^t w))) \\
&\leq \phi(\delta(O_{f,g}(w))),
\end{aligned}$$

which means that

$$\delta(O_{f,g}(w)) \leq \phi(\delta(O_{f,g}(w))).$$

From Lemma 1.1, we easily infer that $\delta(O_{f,g}(w)) = 0$. Therefore $w = fw = gw$, that is, the point w is a common fixed point of f and g. The uniqueness of the common fixed point w of f and g follows immediately from (1.3). For any $p \in W$ and $i \in N$, by (2.2), we have

$$\begin{aligned} \max\{d((fg)^i f^a g^b x, (fg)^{i+p} x) : a, b \in \{0,1\}\} \\ \leq \delta(O_{f,g}((fg)^i x)) \leq \phi^{[\frac{i}{k}]}(\delta(O_{f,g}(x))). \end{aligned} \tag{2.5}$$

Letting p tend to infinity in (2.5), by (2.4), we have

$$\max\{d((fg)^i f^a g^b x, w) : a, b \in \{0,1\}\} \leq \phi^{[\frac{i}{k}]}(\delta(O_{f,g}(x))).$$

This completes the proof. □

Taking $\phi(t) = rt$ in Theorem 2.1, we obtain the following:

Corollary 2.2. *Let f, g be commuting mappings from a complete metric sapce (X, d) into itself and fg be closed. Assume that $O_{f,g}(x)$ is bounded for all $x \in X$ and that there exist $m, n, p, q \in N$ and $r \in [0, 1)$ such that*

$$d(f^m g^n x, f^p g^q y) \leq r\delta(O_{f,g}(x, y)) \tag{2.6}$$

for all $x, y \in X$. Then f and g have a unique common fixed point $w \in X$ and $\lim_{i\to\infty}(fg)^i f^a g^b x = w$ for all $x \in X$ and $a, b \in \{0, 1\}$. Moreover,

$$\max\{d((fg)^i f^a g^b x, w) : a, b \in \{0,1\}\} \leq r^{[\frac{i}{k}]}\delta(O_{f,g}(x))$$

for all $i \in N$, where $k = \max\{m, n, p, q\}$.

Remark 2.1. Corollary 2.1 with $m = n$ and $p = q$ extends, improves and unifies Theorem 1 of [8], Theorem 3 of [16] and Theorem 2.1 of [14].

Kim, Kim, Leem and Ume [14] and Ohta and Nikaido [16] proved the following theorems, respectively:

Theorem KKLU. *Let f, g be commuting mappings from a compact metric space (X, d) into itself and fg be closed. If (1.2) holds, then f and g have a unique common fixed point $w \in X$ and $\lim_{i\to\infty}(fg)^i x = w$ for all $x \in X$.*

Theorem ON. *Let f be a continuous mappings from a compact metric space (X, d) into itself. Assume that there exists $k \in W$ such that*

$$d(f^k x, f^k y) < \delta(O_f(x, y)) \tag{2.7}$$

for all distinct $x, y \in X$. Then f has a unique fixed point $w \in X$ and $\lim_{n\to\infty} f^n x = w$ for all $x \in X$.

First we show, by an example, that (1.2) is not sufficient for the conclusions of Theorem KKLU.

Example 2.1. Let $X = \{0, 1\}$ with the usual metric d. Define $f, g : X \to X$ by $f(0) = g(1) = 1$ and $f(1) = g(0) = 0$. Then (X, d) is a compact metric space, f is continuous and $fg = gf = f$. The continuity of f ensures that f is closed. Taking $m = 1$ and $n = 2$, then we have

$$d(fgx, (fg)^2 y) = 0 < 1 = \delta(O_{f,g}(x, y))$$

for all distinct $x, y \in X$. Thus all the conditions of Theorem KKLU are satisfied. But f and g have no common fixed point in X.

Next we point out that Theorem ON is meaningless for $k = 0$. Suppose that $\delta(X) > 0$. Since X is compact, there exist $x, y \in X$ with $\delta(X) = d(x, y)$. For $k = 0$, by (2.7), we have

$$\delta(X) = d(x, y) < \delta(O_{f,g}(x, y)) \leq \delta(X),$$

which is a contradiction. Thus X is a singleton for $k = 0$.

Now we establish the following result which is a correction of Theorem KKLU and Theorem ON.

Theorem 2.3. *Let f, g be commuting mappings from a compact metric space (X, d) into itself and gf be closed. If (1.4) holds, then f and g have a unique common fixed point $w \in X$, which is also a unique common fixed point of H_{fg}. Moreover, $\lim_{i\to\infty}(fg)^i f^a g^b x = w$ for all $x \in X$ and $a, b \in \{0, 1\}$.*

Proof. Let $A = \bigcap_{i\in N}(fg)^i(X)$. It follows from Lemma 1.2 that A is a nonempty compact subset of X and tht $fg(A) = A$. In virtue of $f(A) \subset A$ and $g(A) \subset A$, we have $A = fg(A) = gf(A) \subset g(A) \subset A$ and so $A = g(A)$. Similarly $A = f(A)$. Thus $f^m g^n(A) = A = f^p g^q(A)$. We claim that A is a singleton. If not, then $\delta(A) > 0$. Obviously there exist $a, b, x, y \in A$ with $\delta(A) = d(a, b), a = f^m g^n x$ and $b = f^p g^q y$. Using (1.4) and Lemma 1.2, we obtain

$$\delta(A) = d(f^m g^n x, f^p g^q y) < \delta\left(\bigcup_{h \in H_{fg}} h(O_{f,g}(x, y))\right)$$
$$\leq \delta\left(\bigcup_{h \in H_{fg}} h(A)\right) \leq \delta(A),$$

which is a contradiction. Hence $A = \{w\}$ for some $w \in X$ and $fw = gw = w$. Suppose that u is also a common fixed point of f and g. Then $u = (fg)^n u$ for all $n \in N$ and hence $u \in A = \{w\}$, which means that $u = w$, that is, w is a unique common fixed point of f and g. It is easy to verify that w is a unique common fixed point of H_{fg}. Moreover, Lemma 1.2 ensures that

$$d((fg)^i f^a g^b x, w) \leq \delta((fg)^i X) \to \delta(A) = 0 \quad \text{as } i \to \infty,$$

where $a, b \in \{0, 1\}$. Consequently, $\lim_{i \to \infty} (fg)^i f^a g^b x = w$ for $a, b \in \{0, 1\}$. This completes the proof. $\square$

From Lemma 1.2 and Theorem 2.3, we have the following:

Corollary 2.4. *Let f, g be commuting mappings from a compact metric space (X, d) into itself and gf be closed. Assume that there exist $m, n, p, q \in W$, $m + p$, $n + q \in N$ such that*

$$d(f^m g^n x, f^p g^q y) < \delta\left(\bigcup_{h \in C_{fg}} h(O_{f,g}(x, y))\right) \tag{2.8}$$

for all $x, y \in X$ with $f^m g^n x \neq f^p g^q y$. Then f and g have a unique common fixed point $w \in X$, which is also a unique common fixed point of C_{fg}. Moreover, $\lim_{i \to \infty} (fg)^i f^a g^b x = w$ for all $x \in X$ and $a, b \in \{0, 1\}$.

Remark 2.2. Corollary 2.4 extends, improves and unifies Theorem 6 of [1], Theorem 4 of [2], Theorem 4 of [3], Theorem 9 of [4], Theorem 2 of [5], Theorem 4 of [7], Theorem 5 of [9], Theorem 2 of [10] and Theorem 4.2 of [12].

We provide some examples to demonstrate that the hypotheses of Theorems 2.1 and 2.3, Corollaries 2.2 and 2.4 are necessary.

Example 2.2. Let $X = [1, +\infty)$ with the usual metric d. Define mappings $f, g : X \to X$ by $fx = 2x$ and $gx = 3x$ for all $x \in X$, respectively. Then (X, d) is a complete metric space, $fg = gf$ and fg is closed. Furthermore, (1.3) and (2.6) are satisfied with $m = p = 1$, $n = q = 2$, $\phi(t) = \frac{1}{2}t$ and $r = \frac{1}{2}$. All of the conditions of Theorem 2.1 and Corollary 2.2 are satisfied except for the boundedness assumption, but f and g have no common fixed point in X.

Example 2.3. Let $X = [0, 1]$ with the usual metric d. Define mappings $f, g : X \to X$ by $fx = \frac{1}{4}(x^3 + 3)$ and $gx = (1 - x)^{\frac{1}{3}}$ for all $x \in X$, respectively. Then (X, d) is a compact metric space, $fgx = \frac{1}{4}(4 - x)$ is closed and $fg(1) = \frac{3}{4} \neq 0 = gf(1)$. Take $m = n = p = q = 1$, $\phi(t) = \frac{1}{2}t$ and $r = \frac{1}{2}$. It is easy to verify that f and g satisfy the following:

$$d(f^m g^n x, f^p g^q y) = \frac{1}{4}|x - y| \leq \frac{1}{2}|x - y| \leq \phi(\delta(O_{f,g}(x, y)))$$

for all $x, y \in X$ and

$$d(f^m g^n x, f^p g^q y) = \frac{1}{4}|x - y| < \frac{1}{2}|x - y| \leq \delta(O_{f,g}(x, y))$$

for all $x, y \in X$ with $f^m g^n x \neq f^p g^q y$. Thus, the conditions of Theorems 2.1 and 2.3, Corollaries 2.2 and 2.4 are satisfied except for the commutativity assumption. But f and g however have no common fixed point in X.

Example 2.4. Let $X = [0, 1]$ with the usual metric d. Define mappings $f, g : X \to X$ by $f(0) = 1$, $fx = \frac{1}{3}x$ for $x \in (0, 1]$ and $g = f^2$. Then f and g are commuting and (X, d) is a compact metric space. Take $m = n = p = q = 1$, $\phi(t) = \frac{1}{2}t$ and $r = \frac{1}{2}$. Since $\lim_{i\to\infty} \frac{1}{i} = \lim_{i\to\infty} \frac{1}{27i} = \lim_{i\to\infty} fg(\frac{1}{i}) = 0$ and $0 \neq \frac{1}{9} = fg(0)$, so fg is not closed. For $x, y \in (0, 1]$, we have

$$d(fgx, fgy) = \frac{1}{27}|x - y| \leq \frac{1}{6}|x - y| \leq \frac{1}{2}\delta(O_{f,g}(x, y)).$$

For $x = 0, y \in [0, 1]$, we have

$$d(fgx, fgy) \leq \frac{1}{9}\left|1 - \frac{1}{3}y\right| < \frac{1}{2} = \frac{1}{2}\delta(O_{f,g}(x, y)).$$

Thus, the conditions of Theorems 2.1 and 2.3, Corollaries 2.2 and 2.4 are satisfied except for the closedness assumption. But f and g have no common fixed point in X.

Theorem 2.5. *Let f, g be closed mappings from a compact metric space (X,d) into itself. Assume that there exist $m,n \in N$ such that*

$$d(f^m x, g^n y) < \delta\Bigg(\bigcup_{h\in H_f} h(O_f(x)), \bigcup_{t\in H_g} t(O_g(y))\Bigg) \tag{2.9}$$

for all $x,y \in X$ with $f^m x \neq g^n y$. Then f and g have a unique common fixed point $w \in X$, which is also a unique common fixed point of H_f and H_g. Moreover, $\lim_{i\to\infty} f^i x = \lim_{i\to\infty} g^i x = w$ for all $x \in X$.

PROOF. Put $A = \bigcap_{i\in N} f^i(X)$ and $B = \bigcap_{i\in N} g^i(X)$. In view of Lemma 1.2, we have $f(A) = A \neq \emptyset$, $g(B) = B \neq \emptyset$ and A and B are compact. Thus there exist $a,x \in A$ and $b,y \in B$ with $d(a,b) = \delta(A,B)$, $a = f^m x$ and $b = g^n y$. We assert that $\delta(A,B) = 0$. If not, by (2.9) and Lemma 1.2, we have

$$d(f^m x, g^n y) < \delta\Bigg(\bigcup_{h\in H_f} h(O_f(x)), \bigcup_{t\in H_g} t(O_g(y))\Bigg) \leq \delta(A,B).$$

Thus we have

$$\delta(A,B) = d(a,b) = d(f^m x, g^n y) < \delta(A,B),$$

which is a contradiction. Therefore $\delta(A,B) = 0$ and there is some $w \in X$ with $A = B = \{w\}$. It is clear that $fw = gw = w$. The rest of the proof is identical with the proof of Theorem 2.3. This completes the proof. □

Remark 2.3. Theorem 2.3 contains Theorem 2.5 of [15] as a special case.

3. Remarks on fixed point theorems of Rehman and Ahmad

In [18], REHMAN and AHMAD proved the following:

Theorem RA1. *Let (X,d) be a complete metric space and $S,T : X \to CB(X)$ satisfy the following:*

$$H(Sx,Ty) \leq k\{D(x,Sx)D(y,Ty)\}^{\frac{1}{2}} \tag{3.1}$$

for all $x,y \in X$, where $k \in (0,1)$. Then S and T have a unique common fixed point $w \in X$.

Corollary RA2. *Let (X, d) be a compact metric space and $S, T : X \to CL(X)$ satisfy* (3.1)*. If S or T is continuous, then S or T has a fixed point in X.*

Theorem RA3. *Let (X, d) be a complete metric space and $S, T : X \to B(X)$ satisfy the following:*

$$\delta(Sx, Ty) \leq k\{H(x, Sx)H(y, Ty)\}^{\frac{1}{2}} \tag{3.2}$$

for all $x, y \in X$, where $k \in [0, 1)$. Then S and T have a unique common fixed point $w \in X$.

Corollary RA4. *Let (X, d) be a compact metric space and $S, T : X \to CL(X)$ satisfy* (3.2)*. If S or T is continuous, then S or T has a fixed point in X.*

Corollary RA5. *Let (X, d) be a compact metric space and $S, T : X \to B(X)$ satisfy* (3.2)*. Then S and T have a unique common fixed point $u \in X$ and $Su = Tu = \{u\}$.*

Theorem RA6. *Let $\{T_n\}_{n \in N}$ be a sequence of self mappings of a complete metric space (X, d). If there exists a constant h such that, for all $x, y \in X$ and $i, j \in N$, $i, j \in N$*

$$d(T_i x, T_j y) \leq h\{d(x, T_i x)d(y, T_j y)\}^{\frac{1}{2}}, \tag{3.3}$$

for some $h \in (0, 1)$, then $\{T_n\}_{n \in N}$ has a unique common fixed point $w \in X$.

Theorem RA7. *Let (X, d) and (X, d') be metric spaces satisfying the following:*

(i) *$d(x, y) \leq d'(x, y)$ for all $x, y \in X$;*

(ii) *X is complete with respect to d;*

(iii) *$f, g : X \to X$ are self-mappings such that f is continuous with respect to d and, for all $x, y \in X$,*

$$d'(fx, fy) \leq h\{d'(x, fx)d'(y, gy)\}^{\frac{1}{2}} \tag{3.4}$$

for some $h \in [0, 1)$. Then f and g have a unique common fixed point $w \in X$.

We assert, by the following results, that Theorems RA1, RA3, RA6, RA7 and Corollaries RA2$\sim$RA5 are meaningless.

Theorem 3.1. *Let (X, d), S, T and w be as in Theorem RA1. Then $w \in Sw = Tw = Sx = Tx$ for all $x \in X$.*

Proof. For any $x \in X$, by (3.1) and Theorem RA1, we have

$$H(Tx, Sw) \leq k\{D(x, Tx)D(w, Sw)\}^{\frac{1}{2}} = 0,$$

which implies that $Tx = Sw$. Similarly, $Sx = Tw$. Therefore $w \in Sw = Tw = Sx = Tx$ for all $x \in X$. This completes the proof. □

Corollary 3.2. *Let (X, d) be a compact metric space and $S, T : X \to CL(X)$ satisfy (3.1). Then there exists $w \in X$ such that $w \in Sw = Tw = Sx = Tx$ for all $x \in X$.*

Proof. Since (X, d) is a compact metric space, $CL(X) = CB(X)$. Thus Corollary 3.2 follows from Theorem 3.1. □

Theorem 3.3. *Let (X, d), S and T be as in Theorem RA3. Then there exists $w \in X$ such that $Sx = Tx = \{w\}$ for all $x \in X$.*

Proof. It is easy to verify that (3.2) is equivalent to the following:

$$\delta(Sx, Ty) \leq k\{\delta(x, Sx)\delta(y, Ty)\}^{\frac{1}{2}} \tag{3.5}$$

for all $x, y \in X$, where $k \in [0, 1)$. We claim that there exists $w \in X$ such that $\delta(w, Sw)\delta(w, Tw) = 0$. Otherwise, $\delta(x, Sx)\delta(x, Tx) > 0$ for all $x \in X$. We consider the following two cases:

Case 1. Suppose that $k = 0$. (3.5) implies that $Sx = Ty$ for all $x, y \in X$. Take $x \in X$ and $y \in Sx$. Then $\delta(y, Ty) = \delta(y, Sx) = 0$, which is a contradiction.

Case 2. Suppose that $k \in (0, 1)$. Take $x_0 \in X$ and select a sequence $\{x_n\}_{n \in N}$ in X such that $x_{2n+1} \in Sx_{2n}$ for $n \in W$ and $x_{2n} \in Tx_{2n-1}$ for $n \in N$. (3.5) ensures that

$$\delta(x_{2n}, Sx_{2n}) \leq \delta(Sx_{2n}, Tx_{2n-1}) = k\{\delta(x_{2n}, Sx_{2n})\delta(x_{2n-1}, Tx_{2n-1})\}^{\frac{1}{2}},$$

which implies that

$$\delta(x_{2n}, Sx_{2n}) \leq r\delta(x_{2n-1}, Tx_{2n-1}),$$

where $r = k^2 \in (0,1)$. Similarly, $\delta(x_{2n-1}, Tx_{2n-1}) \le r\delta(x_{2n-2}, Sx_{2n-2})$. Set $M = \max\{\delta(x_2, Sx_2), \delta(x_1, Tx_1)\}$. For $m > n \ge 2$, we have

$$\begin{aligned}(3.6)\quad &\max\{\delta(x_{2n}, Sx_{2n}), \delta(x_{2n-1}, Tx_{2n-1})\}\\ &\qquad \le r^2 \max\{\delta(x_{2n-2}, Sx_{2n-2}), \delta(x_{2n-3}, Tx_{2n-3})\} \le r^{2(n-1)}M\end{aligned}$$

and

$$(3.7)\qquad d(x_n, x_m) \le \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \le \sum_{i=n}^{m-1} r^{i-2}M \le \frac{r^{n-2}}{1-r}M.$$

Since $r \in (0,1)$ and (X,d) is complete, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X and so it converges to some $w \in X$. Using (3.5), we have

$$\begin{aligned}\delta(w, Sw) &\le d(w, x_{2n+2}) + \delta(Sw, x_{2n+2})\\ &\le d(w, x_{2n+2}) + \delta(Sw, Tx_{2n+1})\\ &\le d(w, x_{2n+2}) + k\{\delta(w, Sw)\delta(x_{2n+1}, Tx_{2n+1})\}^{\frac{1}{2}}.\end{aligned}$$

Letting n tend to infinity, by (3.6), (3.7) and boundedness of S, we immediately obtain $\delta(w, Sw) = 0$, which is also a contradiction.

Consequently, $\delta(w, Sw)\delta(w, Tw) = 0$ for some $w \in X$. We assume without loss of generality, that $\delta(w, Sw) = 0$, that is, $Sw = \{w\}$. Using (3.5), for any $x \in X$, we have

$$\delta(w, Tx) = \delta(Sw, Tx) \le k\{\delta(w, Sw)\delta(x, Tx)\}^{\frac{1}{2}} = 0,$$

which implies that $Tx = \{w\}$. Clearly $Tw = \{w\} = Tx = Sw$. On the other hand, by (3.5), we have

$$\delta(w, Sx) = \delta(Sx, Tw) \le k\{\delta(x, Sx)\delta(w, Tw)\}^{\frac{1}{2}} = 0.$$

Therefore, $Sx = \{w\} = Tx$. This completes the proof. □

Corollary 3.4. *Let (X,d) be a compact metric space and $S, T : X \to B(X)$ satisfy* (3.2)*. Then there exists $w \in X$ such that $\{w\} = Sx = Tx$ for all $x \in X$.*

Proof. It follows from the compactness of X that $CL(X) \subset B(X)$. Thus Corollary 3.4 follows from Theorem 3.3. □

Theorem 3.5. *Let (X, d), $\{T_n\}_{n\in N}$ and w be as in Theorem RA6. Then $T_n x = w$ for all $x \in X$ and $n \in N$.*

Proof. Theorem RA6 ensures that $T_n w = w$ for all $n \in N$. By (3.2), for all $x \in X$ and $i, j \in N$, we have

$$d(w, T_j x) = d(T_i w, T_j x) \le h\{d(w, T_i w) d(x, T_j x)\}^{\frac{1}{2}} = 0,$$

which implies that $T_j x = w$. This completes the proof. □

Theorem 3.6. *Let $(X, d), (X, d'), f, g$ and w be as in Theorem RA7. Then $fx = gx = w$ for all $x \in X$.*

Proof. In view of (3.3) and Theorem RA7, we have

$$d'(w, gx) = d'(fw, gx) \le h\{d'(w, fw) d'(x, gx)\}^{\frac{1}{2}} = 0,$$

which implies that $w = gx$ for all $x \in X$. Similarly, $fx = w$ for all $x \in X$. This completes the proof. □

Acknowledgement. The authors thank the referee for his valuable suggestions for the improvement of this paper.

References

[1] B. Fisher, Results and a conjecture on fixed points, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **92** (1977), 150–153.

[2] B. Fisher, On three fixed point mappings for compact metric spaces, *Indian J. Pure Appl. Math.* **8** (1977), 479–481.

[3] B. Fisher, Some theorems on fixed points, *Studia Sci. Math. Hungarica* **12** (1977), 159–160.

[4] B. Fisher, Theorems on fixed points, *Riv. Mat. Univ. Param.* **4** (1978), 109–114.

[5] B. Fisher, A fixed point theorem for compact metric spaces, *Publ. Math. Debrecen* **25** (1978), 193–194.

[6] B. Fisher, Results on common fixed points on bounded metric spaces, *Math. Sem. Notes, Kobe Univ.* **7** (1979), 73–80.

[7] B. Fisher, Quasi-contractions on metric spaces, *Proc. Amer. Math. Soc.* **75** (1979), 321–325.

[8] B. Fisher, Results on common fixed points on complete metric spaces, *Glasgow Math. J.* **21** (1980), 165–167.

[9] B. Fisher, Common fixed points of commuting mappings, *Bull. Inst. Math. Acad. Sinica* **9** (1981), 399–406.

[10] B. Fisher, A common fixed point theorem for four mappings on a compact metric space, *Bull. Inst. Math. Acad. Sinica* **12** (1984), 249–252.

[11] B. Fisher and S. Sessa, A fixed point theorem for commuting mappings, Non-linear Functional Analysis and its Application: Mathematical and Physical Sciences, vol. 173, *Reidel Publishing Company*, 1986, 223–226.

[12] G. Jungck, Common fixed points for commuting and compatible maps on compacta, *Proc. Amer. Math. Soc.* **103** (1988), 977–983.

[13] K. Kim and K. H. Leem, Note on common fixed point theorems in metric spaces, *Comm. Korean Math. Soc.* **11** (1996), 109–115.

[14] K. Kim, T. H. Kim, K. H. Leem and J. S, Ume, Common fixed point theorems relating to the diameter of orbits, *Math. Japonica* **47** (1998), 103–108.

[15] Z. Liu, Extensions of a fixed point theorem of Gerald Jungck, *Chinese J. Math. (Taiwan)* **21** (1993), 159–164.

[16] M. Ohta and G. Nikaido, Remarks and fixed point theorems in complete metric spaces, *Math. Joponica* **39** (1994), 287–296.

[17] S. Park, On general contractive type conditions, *J. Korean Math. Soc.* **17** (1980), 131–140.

[18] F. U. Rehman and B. Ahmad, Some fixed point theorems in complete metric spaces, *Math. Japonica* **36** (1991), 239–243.

[19] B. E. Rhoades, A comparison of various definitions of contractive mappings, *Trans. Amer. Math. Soc.* **226** (1977), 259–290.

[20] S. P. Singh and B. A. Meade, On common fixed point theorems, *Bull. Austral. Math. Soc.* **16** (1977), 49–53.

[21] M. R. Tasković, Some results in the fixed point theory II, *Publ. Inst. Math.* **27** (1980), 249–258.

[22] M. R. Tasković, Some new principles in fixed point theory, *Math. Japonica* **35** (1990), 645–666.

ZEQING LIU
DEPARTMENT OF MATHEMATICS
LIAONING NORMAL UNIVERSITY
DALIAN, LIAONING 116029
P.R. CHINA

YUGUANG XU
DEPARTMENT OF MATHEMATICS
KUNMING JUNIOR NORMAL COLLEGE
KYNMING, YUNNAN 650031
P.R. CHINA

YEOL JE CHO
DEPARTMENT OF MATHEMATICS
GYEONGSANG NATIONAL UNIVERSITY
CHINJU 660–701
KOREA

E-mail: yjcho@nongae.gsnu.ac.kr

(Received July 17, 2001; revised April 16, 2002)