

Conservative semisprays on Finsler manifolds

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Abstract. In this paper we present a general theory of conservative torsion-free horizontal endomorphisms (nonlinear connections) on a Finsler manifold (M, E) . Since their torsion vanishes these endomorphisms can always be written in the form

$$h = \frac{1}{2}(1 + [J, S]),$$

where S is a semispray on M . This means that all of problems can be formulated in terms of semisprays as well. For example their existence problem will be completely solved including a representative process to construct such kind of horizontal endomorphisms. Moreover, putting a positive definite two-dimensional Finsler manifold (or special types of Finsler manifolds such as Riemannian and Randers manifolds of dimension n) we characterize all of them under the condition $\operatorname{div} S = 0$.

1. Introduction

In their work [8] the authors constructed special Finsler connections on a Finsler manifold starting out from torsion-free conservative horizontal endomorphisms. Among others it was proved that for any conservative torsion-free horizontal endomorphism h there exists a unique Finsler connection (D, h) on M such that

- (i) D is metrical;
- (ii) the $(v)v$ -torsion of D vanishes;
- (iii) the $(h)h$ -torsion of D vanishes.

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As we can see in [8] the rules of calculation with respect to these connections are formally the same as those with respect to the classical Cartan connection. Moreover, adding a further condition to ones above we can get it back yet. Of course this is not the only example of such kind of Finsler connections. It is well-known (see e.g. [9]) that the so-called *Wagner connections* have a similar character. Actually we can say that a Wagner connection is a “*Cartan connection with nonvanishing (h)h-torsion*”, i.e. it is a *generalized Cartan connection*. A necessary and sufficient condition for a Wagner connection to coincide with the classical Cartan connection is just that the Wagner endomorphism arises from a semispray, i.e. its torsion vanishes. Due to an explicit relation between the (canonical) Barthel endomorphism of (M, E) and a Wagner endomorphism the problem of existence is solved. We can easily construct Wagner connections on a Finsler manifold starting out from a smooth function $\alpha \in C^\infty(M)$. However, apart from the Barthel endomorphism the analogous problem is open in case of conservative torsion-free horizontal endomorphisms.

The purpose of this paper are the following:

- to give a necessary and sufficient condition for a torsion-free horizontal endomorphism to be conservative;
- to give a process to construct such kind of horizontal endomorphisms;
- to characterize them in case of two-dimensional Finsler manifolds (or Riemannian and Randers manifolds of dimension n) under the condition $\operatorname{div} S = 0$.

(We emphasize again that these questions can also be formulated in terms of semisprays!)

1. Preliminaries

1.1. Throughout the paper we use the terminology and conventions described in [8] (see also [2], [3] and [9]). Now we briefly summarize the basic notation.

- (i) M is an n (> 1)-dimensional, C^∞ , connected, paracompact manifold, $C^\infty(M)$ is the ring of real-valued smooth functions on M .
- (ii) $\pi : TM \rightarrow M$ is the tangent bundle of M , $\pi_0 : TM \rightarrow M$ is the bundle of nonzero tangent vectors.

- (iii) $\mathfrak{X}(M)$ denotes the $C^\infty(M)$ -module of vector fields on M .
- (iv) $\Omega^k(M)$ ($k \in \mathbb{N}^+$) is the module of (scalar) k -forms on M , $\Omega^0(M) := C^\infty(M)$.
- (v) $\Psi^k(M)$ ($k \in \mathbb{N}^+$) is the $C^\infty(M)$ -module of vector k -forms on M , $\Psi^0(M) := \mathfrak{X}(M)$.
- (vi) ι_X , $\mathcal{L}_X$ ($X \in \mathfrak{X}(M)$) and d are the *insertion operator*, the *Lie-derivative* (with respect to X) and the *exterior derivative*, respectively.

1.2. We shall apply some simple facts of the Frölicher–Nijenhuis calculus of vector-valued forms. Recall that if $K \in \Psi^1(M)$, $Y \in \mathfrak{X}(M)$ then their *Frölicher–Nijenhuis bracket* $[K, Y] \in \Psi^1(M)$ acts as follows:

$$(1) \quad [K, Y](X) = [K(X), Y] - K[X, Y] \quad (X \in \mathfrak{X}(M)).$$

For the derivation d_K induced by K we have:

$$(2) \quad d_K f = df \circ K \quad (f \in C^\infty(M)).$$

1.3. Vertical apparatus. Semispray, spray. Let us consider the tangent bundle $\pi : TM \rightarrow M$. $\mathfrak{X}^v(TM)$ denotes the $C^\infty(TM)$ -module of vertical vector fields on TM , $C \in \mathfrak{X}^v(TM)$, $J \in \Psi^1(TM)$ are the *Liouville vector field* and *vertical endomorphism*, respectively. We have:

$$(3) \quad \begin{cases} \text{Im } J = \text{Ker } J = \mathfrak{X}^v(TM), \quad J^2 = 0, \\ [C, J] = -J \quad (\text{i.e. } J \text{ is homogeneous of degree } 0), \\ d_J = d_J \circ \mathcal{L}_C - \mathcal{L}_C \circ d_J. \end{cases}$$

The vertical lift of a vector field $X \in \mathfrak{X}(M)$ is denoted by X^v .

Definition. A mapping $S : v \in TM \rightarrow S(v) \in T_v TM$ is said to be a *semispray* on M if it satisfies the conditions:

- (Spr1) S is smooth on TM ,
- (Spr2) $JS = C$.

A semispray is called a *spray* if it is homogeneous of degree 2, i.e.

$$(Spr3) \quad [C, S] = S.$$

(Note that (Spr3) implies for any spray to be a vector field of class C^1 on TM .)

The *vertical* and *complete lifts* of a function $\alpha \in C^\infty(M)$ are given by

$$(4) \quad \alpha^\vee := \alpha \circ \pi, \quad \alpha^c := S\alpha^\vee,$$

where S is an arbitrary semispray on M , respectively. For any $X \in \mathfrak{X}(TM)$

$$(5) \quad J[JX, S] = JX.$$

(For a proof see [2], p. 295.)

Remark 1. In the sequel we shall consider forms over TM or $\mathcal{T}M$. *Differentiability of vector (and scalar) k -forms will be required only over TM , unless otherwise stated.*

1.4. Horizontal endomorphisms ([2], [3] and see also [8])

Definition. A vector 1-form $h \in \Psi^1(TM)$ is said to be a *horizontal endomorphism* on M if the following conditions are satisfied:

(He1) h is smooth over $\mathcal{T}M$,

(He2) h is a projector, i.e. $h^2 = h$,

(He3) $\text{Ker } h = \mathfrak{X}^\vee(TM)$.

The *associated semispray* of h is defined by the formula

$$(6) \quad S_h := h(S),$$

where S is an arbitrary semispray on M . The *tension* of h is the vector 1-form

$$(7) \quad H := [h, C] \in \Psi^1(TM).$$

The vector 2-form

$$(8) \quad t := [J, h] \in \Psi^2(TM)$$

is said to be the *torsion* of h . If $H = 0$, then h is called *homogeneous*.

Remark 2. It is a well-known fact (see e.g. [2]) that a horizontal endomorphism h arises from a semispray, i.e. it has a form

$$(9) \quad h = \frac{1}{2}(1 + [J, S]),$$

if and only if its torsion vanishes.

J and h are obviously related as follows:

$$(10) \quad h \circ J = 0, \quad J \circ h = J$$

and, furthermore, any horizontal endomorphism h determines an *almost complex structure* $F \in \Psi^1(TM)$ ($F^2 = -1$, F is smooth on TM) such that

$$(11) \quad F \circ J = h, \quad F \circ h = -J.$$

(For the details see e.g. [2].)

1.5. Finsler manifolds

Definition. Let a function $E : TM \rightarrow \mathbb{R}$ be given. The pair (M, E) , or simply M , is said to be a *Finsler manifold* with *energy function* E if the following conditions are satisfied:

- (F0) $\forall v \in TM : E(v) > 0, E(0) = 0$,
- (F1) E is of class C^1 on TM and smooth on TM ,
- (F2) $C(E) = 2E$ (i.e. E is homogeneous of degree 2),
- (F3) the *fundamental form* $\omega := dd_J E \in \Omega^2(TM)$ is symplectic.

The mapping

$$(12) \quad \begin{cases} g : \mathfrak{X}^v(TM) \times \mathfrak{X}^v(TM) \rightarrow C^\infty(TM), \\ (JX, JY) \rightarrow g(JX, JY) := \omega(JX, Y) \end{cases}$$

is a well-defined, nondegenerate symmetric bilinear form which is said to be the *Riemann-Finsler metric* of (M, E) . The Finsler manifold is called *positive definite* if g is positive definite.

We have the following important identities:

$$(13) \quad \iota_C \omega = d_J E, \quad \mathcal{L}_C \omega = \omega$$

(i.e. the fundamental form ω is homogeneous of degree 1).

The fundamental lemma of Finsler geometry [2]. On a Finsler manifold (M, E) there is a unique horizontal endomorphism h such that

- (B1) h is conservative, i.e. $d_h E = 0$,
- (B2) h is homogeneous,
- (B3) h is torsion-free, i.e. $t = 0$.

Explicitly,

$$(14) \quad h = \frac{1}{2}(1 + [J, S]),$$

where S is the *canonical spray* defined by the formula

$$(15) \quad \iota_S \omega = -dE.$$

h is called the *Barthel endomorphism* of the Finsler manifold (M, E) .

Let h be an *arbitrary* horizontal endomorphism on M , $\nu := 1 - h$. The mapping

$$(16) \quad \begin{cases} g_h : \mathfrak{X}(TM) \times \mathfrak{X}(TM) \rightarrow C^\infty(TM), \\ (X, Y) \rightarrow g_h(X, Y) := g(JX, JY) + g(\nu X, \nu Y) \end{cases}$$

is a well-defined pseudo-Riemannian metric on TM which is said to be the *prolongation* of g *along* h .

Definition. The tensor field $\mathcal{C}$ satisfying the condition

$$\omega(\mathcal{C}(X, Y), Z) := \frac{1}{2} \mathcal{L}_{JX}(J^* g_h)(Y, Z) \quad (X, Y, Z \in \mathfrak{X}(TM))$$

is called the *first Cartan tensor* of the Finsler manifold.

Remark 3. It is easy to check that $\mathcal{C}$ is independent of the choice of h and

- (i) it is semibasic,
- (ii) its *lowered tensor*

$$\mathcal{C}_\flat(X, Y, Z) := g(\mathcal{C}(X, Y), JZ)$$

is totally symmetric,

- (iii) $\mathcal{C}^\circ := \iota_S \mathcal{C} = 0$ (S is an arbitrary semispray on M).

Let a smooth function $\varphi : TM \rightarrow \mathbb{R}$ (or $\varphi : \mathcal{T}M \rightarrow \mathbb{R}$) be given. Since the fundamental form ω is symplectic, there exists a unique vector field $\text{grad } \varphi \in \mathfrak{X}(TM)$ such that

$$(17) \quad \iota_{\text{grad } \varphi} \omega = d\varphi;$$

this vector field is called the *gradient* of φ .

Lemma 1. *The gradient vector field of a vertical lift $\alpha^\vee := \alpha \circ \pi$ ($\alpha \in C^\infty(M)$) has the following properties:*

- (i) $\text{grad } \alpha^\vee \in \mathfrak{X}^\vee(TM)$, $[C, \text{grad } \alpha^\vee] = -\text{grad } \alpha^\vee$ (i.e. it is homogeneous of degree 0),
- (ii) $\text{grad } \alpha^\vee(E) = \alpha^c$,
- (iii) $\iota_F \text{grad } \alpha^\vee C = -\frac{1}{2}[J, \text{grad } \alpha^\vee]$.

For a proof see [7] and [9]. $\square$

Definition [4]. Let (M, E) be a Finsler manifold and consider the volume form

$$w := \frac{(-1)^{n(n+1)/2}}{n!} \omega^n$$

on TM . The *divergence* of a vector field $X \in \mathfrak{X}(TM)$ is the function $\text{div } X$ given by formula

$$(18) \quad (\text{div } X)w = \mathcal{L}_X w.$$

Lemma 2 [7]. *For any function $\varphi \in C^\infty(TM)$*

$$(19) \quad \text{div}(\text{grad } \varphi) = 0; \quad \text{div } C = n.$$

2. Conservative vector fields on a Finsler manifold

In what follows, we will denote the canonical spray and the Barthel endomorphism of a Finsler manifold (M, E) by S_h and h , respectively. (It is well-known that the canonical spray is just the semispray associated with h .)

Definition. Let (M, E) be a Finsler manifold. A *horizontal endomorphism* $\tilde{h}$ on M is said to be *conservative* (with respect to the energy function E) if

$$d_{\tilde{h}} E = 0.$$

A *semispray* $\tilde{S}$ is called *conservative* if the induced horizontal endomorphism

$$\tilde{h} := \frac{1}{2}(1 + [J, \tilde{S}])$$

is conservative.

A vector field $V \in \mathfrak{X}^v(TM)$ is *conservative* if the semispray

$$\tilde{S} := S_h + V$$

is conservative:

Proposition 1. A vector field $V \in \mathfrak{X}^v(TM)$ is conservative if and only if

$$(20) \quad \iota_V \omega = d_J(VE),$$

or in an equivalent form, a semispray $\tilde{S}$ is conservative if and only if

$$\iota_{\tilde{S}} \omega + dE = d_J(\tilde{S}E).$$

PROOF. For any vector field $X \in \mathfrak{X}(TM)$,

$$\begin{aligned} i_V \omega(X) - d_J(VE)(X) &= \omega(V, X) - JX(VE) = dd_J E(V, X) - JX(VE) \\ &= V(JX(E)) - J[V, X](E) - JX(VE) \\ &= -[JX, V](E) + J[X, V](E) \stackrel{(1)}{=} -[J, V](X)(E). \end{aligned}$$

On the other hand if $\tilde{S} := S_h + V$ and $\tilde{h}$ is the induced horizontal endomorphism then we get:

$$\begin{aligned} d_{\tilde{h}} E(X) &= \tilde{h}(X)(E) = \frac{1}{2}(1 + [J, \tilde{S}])(X)(E) \\ &\stackrel{(14)}{=} h(X)(E) + \frac{1}{2}[J, V](X)(E) \stackrel{(B1)}{=} \frac{1}{2}[J, V](X)(E), \end{aligned}$$

which implies our statement. $\square$

Corollary 1. If $V \in \mathfrak{X}^v(TM)$ is a conservative vector field then the following homogeneity properties are valid:

$$(21) \quad C(VE) = V(E); \quad \mathcal{L}_C(d_J(VE)) = 0; \quad [C, V] = -V.$$

PROOF. From the hypothesis it follows that

$$0 = \frac{1}{2}(1 + [J, S_h + V])(S_h)(E) = h(S_h)(E) + \frac{1}{2}[J, V](S_h)(E)$$

$$\begin{aligned} &\stackrel{(B1)}{=} \frac{1}{2}[J, V](S_h)(E) \stackrel{(1)}{=} \frac{1}{2}[C, V](E) - \frac{1}{2}J[S_h, V](E) \\ &\stackrel{(5)}{=} \frac{1}{2}[C, V](E) + \frac{1}{2}V(E). \end{aligned}$$

So we get

$$\begin{aligned} 0 &= [C, V](E) + V(E) = C(VE) - V(CE) + V(E) \\ &\stackrel{(F2)}{=} C(VE) - V(E) \Rightarrow C(VE) = V(E). \end{aligned}$$

Consequently,

$$\mathcal{L}_C(d_J(VE)) \stackrel{(3)}{=} d_J\mathcal{L}_C(VE) - d_J(VE) = d_J(VE) - d_J(VE) = 0.$$

Finally,

$$\begin{aligned} \iota_{[C, V]} \omega &= \mathcal{L}_C \iota_V \omega - \iota_V \mathcal{L}_C \omega \stackrel{(13)}{=} \mathcal{L}_C \iota_V \omega - \iota_V \omega \\ &\stackrel{(20)}{=} \mathcal{L}_C(d_J(VE)) - \iota_V \omega = -\iota_V \omega \Rightarrow [C, V] = -V. \quad \square \end{aligned}$$

Corollary 2. *Let $\tilde{S}$ be a conservative semispray; $V := \tilde{S} - S_h$. Then*

$$[C, \tilde{S}] = \tilde{S} - 2V,$$

i.e. the deviation of $\tilde{S}$ is just $-2V$.

PROOF. Using the homogeneity property

$$[C, S_h] = S_h$$

of the canonical spray, an easy calculation shows that

$$[C, \tilde{S}] - \tilde{S} = [C, V] - V \stackrel{(21)}{=} -2V. \quad \square$$

Corollary 3. *Consider a torsion-free conservative horizontal endomorphism $\tilde{h}$ given by the formula*

$$\tilde{h} = \frac{1}{2}(1 + [J, \tilde{S}]),$$

where $\tilde{S}$ is a conservative semispray; $V := \tilde{S} - S_h$.

Then

- (i) the associated semispray of $\tilde{h}$ coincides with the canonical spray S_h ,
- (ii) $\tilde{H} = [J, V]$.

PROOF. It is well-known (see [2]) that the associated semispray of $\tilde{h}$ is just

$$\tilde{S} + \frac{1}{2}\tilde{S}^*,$$

where $\tilde{S}^*$ denotes the deviation of $\tilde{S}$. Compare this relation with Corollary 2 we get our first statement.

Using GRIFONE's decomposition formula (see also [2]) (i) implies that

$$\tilde{h} = h + \frac{1}{2}\tilde{H}.$$

On the other hand

$$\tilde{h} = \frac{1}{2}(1 + [J, \tilde{S}]) = \frac{1}{2}(1 + [J, S_h]) + \frac{1}{2}[J, V] = h + \frac{1}{2}[J, V],$$

and thus

$$\tilde{H} = [J, V]. \quad \square$$

Remark 4. The converse of the previous statement is obviously true: If $\tilde{h}$ is a torsion-free horizontal endomorphism on M such that

- (i) the associated semispray of $\tilde{h}$ coincides with the canonical semispray S_h , i.e. $S_{\tilde{h}} = S_h$,
- (ii) the tension of $\tilde{h}$ can be written in the form

$$\tilde{H} = [J, V],$$

where $V \in \mathfrak{X}^v(TM)$ is a conservative vector field, then $\tilde{h}$ is also conservative.

(Use Grifone's decomposition formula to prove this observation!)

Theorem 1. Let (M, E) be a Finsler manifold and suppose that the function $\varphi \in C^\infty(TM)$ is homogeneous of degree 1, i.e. $C(\varphi) = \varphi$.

Then the vector field $V \in \mathfrak{X}^v(TM)$ defined by the formula

$$(22) \quad \iota_V \omega = d_J \varphi$$

is conservative.

PROOF. It is clear from the definition that V is a vertical vector field. Substituting the canonical spray S_h into (22) we get

$$\iota_V \omega(S_h) = \omega(V, S_h) = -\omega(S_h, V) = -\iota_{S_h} \omega(V) \stackrel{(15)}{=} V(E).$$

On the other hand

$$d_J \varphi(S_h) \stackrel{(2)}{=} J(S_h)(\varphi) \stackrel{(\text{Spr}2)}{=} C(\varphi) = \varphi,$$

using the homogeneity property of φ . This means that the vector field V satisfies the relation

$$\iota_V \omega = d_J(V E),$$

i.e., by Proposition 1, V is conservative. $\square$

Definition. Let (M, E) be a Finsler manifold. The mapping

$$\Delta : C^\infty(TM) \rightarrow C^\infty(TM)$$

$$\varphi \rightarrow \Delta \varphi := -\operatorname{div}(J \operatorname{grad} \varphi)$$

is called the *Brickell operator* of the Finsler manifold (M, E) .

(A nice application of the elliptic differential operator Δ can be found in BRICKELL's paper [1].)

Remark 5. Let $(U, (u^i)_{i=1}^n)$ be a chart on M and consider the induced chart

$$(\pi^{-1}(U), (x^i, y^i)_{i=1}^n); \quad x^i := u^i \circ \pi,$$

$$y^i : v \in \pi^{-1}(U) \rightarrow y^i(v) := v(u^i) \quad (1 \leq i \leq n)$$

of the tangent manifold TM . Then we get the following coordinate expression:

$$(23) \quad J \operatorname{grad} \varphi = -g^{ij} \frac{\partial \varphi}{\partial y^i} \frac{\partial}{\partial y^j},$$

where

$$g_{ij} := g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = \frac{\partial^2 E}{\partial y^i \partial y^j}; \quad (g^{ij}) = (g_{ij})^{-1}.$$

$$(24) \quad \Delta\varphi = g^{ij} \frac{\partial^2 \varphi}{\partial y^i \partial y^j}.$$

(The proof is an easy straightforward calculation based on the definitions and [4].)

Lemma 3. *Let a function $\varphi \in C^\infty(TM)$ be given and suppose that φ is homogeneous of degree 0. Then the following assertions are equivalent:*

- (i) $\varphi = \alpha^\vee$ ($\alpha \in C^\infty(M)$), i.e. φ is a vertical lift;
- (ii) $\Delta\varphi \geq 0$ (or $\Delta\varphi \leq 0$).

PROOF. Since φ is homogeneous of degree 0, it attains a maximum (or a minimum) on each fiber $T_p M$ ($p \in M$). So the nontrivial implication (ii) $\implies$ (i) is a special case of Hopf's strong maximum principle (see [11], p. 26). $\square$

Proposition 2. *If $V \in \mathfrak{X}^\vee(TM)$ is a conservative vector field, then*

$$(25) \quad J \operatorname{grad}(VE) = -V$$

and, consequently,

$$(26) \quad \Delta(VE) = \operatorname{div} V.$$

PROOF. For any vector field $X \in \mathfrak{X}(TM)$,

$$\begin{aligned} \omega(J \operatorname{grad}(VE), X) &\stackrel{(12)}{=} g(J \operatorname{grad}(VE), JX) = g(JX, J \operatorname{grad}(VE)) \\ &\stackrel{(12)}{=} \omega(JX, \operatorname{grad}(VE)) = -\omega(\operatorname{grad}(VE), JX) \\ &\stackrel{(17)}{=} -JX(VE) = -d_J(VE)(X) \stackrel{(20)}{=} -\iota_V \omega(X), \end{aligned}$$

from which it follows that

$$J \operatorname{grad}(VE) = -V. \quad \square$$

Proposition 3. *The normalized Liouville vector field $C_0 := L^{-1}C$, where the function L is defined by the formula $E = \frac{1}{2}L^2$, is conservative.*

Then

$$\Delta(C_0 E) = (n-1) \frac{1}{L}.$$

(Note that the function L is uniquely determined up to the sign!)

PROOF. From (F2) we get immediately the homogeneity property

$$C(L) = L,$$

i.e. L is homogeneous of degree 1. According to Theorem 1, this means that the vector field $V \in \mathfrak{X}^v(\mathcal{T}M)$ defined by the formula

$$i_V \omega = d_J L$$

is conservative.

Since

$$\iota_C \omega \stackrel{(13)}{=} d_J E = d_J \left(\frac{1}{2} L^2 \right) = L d_J L = L \iota_V \omega,$$

we get the relation

$$C = LV \Rightarrow V = \frac{1}{L} C.$$

On the other hand

$$\begin{aligned} \Delta(C_0 E) &\stackrel{(26)}{=} \operatorname{div} C_0 = \operatorname{div} \left(\frac{1}{L} C \right) \\ &\stackrel{(18)}{=} \frac{1}{L} \operatorname{div} C + C \left(\frac{1}{L} \right) \stackrel{\text{Lemma 2}}{=} n \frac{1}{L} - \frac{1}{L^2} C(L) \\ &= (n-1) \frac{1}{L}. \end{aligned} \quad \square$$

Remark 6. Since the normalized Liouville vector field can be considered as a *canonical* object of a Finsler manifold (M, E) it seems to be actually the most important example of conservative vector fields. Moreover, by the help of C_0 , we can construct further canonical ones on (M, E) .

For example, an easy calculation shows that the torsion-free conservative horizontal endomorphism h_0 induced by the conservative semispray

$$S_0 := S_h + C_0$$

and the Barthel endomorphism are related as follows:

$$(27) \quad h_0 = h + \frac{1}{2L} \left(J - \frac{1}{L} d_J L \otimes C \right).$$

Denote $\mathcal{C}'$ and $\mathcal{C}'_0$ the second Cartan tensors belonging to h and h_0 , respectively (for the definition see [8]). Then for any vector fields $X, Y \in \mathfrak{X}(\mathcal{T}M)$,

$$(28) \quad \begin{aligned} \mathcal{C}'_0(X, Y) = & \mathcal{C}'(X, Y) + \frac{1}{2L} \left\{ \mathcal{C}(X, Y) - \frac{1}{L} JX(L)JY - \frac{1}{L} JY(L)JX \right\} \\ & + \frac{1}{L^2} \left[\frac{1}{L} JX(L)JY(L) - \frac{1}{2} dd_J L(JX, Y) \right] C. \end{aligned}$$

Using Theorems 4.3, 4.5 and 4.9 of our stimulating paper [8], pp. 46–51, these formulas allow one to describe some new (more or less canonical) Finsler connections on the Finsler manifold (M, E) :

- A *Berwald-type Finsler connection* associated with h_0 (see 4.3. Theorem in [8], p. 47);
- A *Cartan-type Finsler connection* associated with h_0 (see 4.5. Theorem in [8], p. 47);
- A *Chern–Rund-type Finsler connection* associated with h_0 (see 4.9. Theorem in [8], p. 50);
- etc. (see for example a *Hashiguchi-type Finsler connection* associated with h_0 ; [6]).

(It seems to be an important application of our results to the theory of Finsler connections.)

By using Theorem 1, the following observations can be easily verified:

- (i) For any vector field $X \in \mathfrak{X}(M)$ the vertical lift $X^\vee$ is conservative. Moreover, the induced horizontal endomorphism coincides with the Barthel endomorphism.

(ii) For any function $\alpha \in C^\infty(M)$ the gradient vector field $\text{grad } \alpha^\vee$ is conservative. Then the induced horizontal endomorphism h_α and the Barthel endomorphism are related as follows

$$h_\alpha = h - \iota_F \text{grad } \alpha^\vee \mathcal{C}.$$

(Use Lemma 1 to derive this relation; see [10], pp. 25–26.)

(iii) The conservative vector fields form a $C^\infty(M)$ -module by the “scalar” multiplication

$$C^\infty(M) \times \mathfrak{X}^\vee(TM) \rightarrow \mathfrak{X}^\vee(TM),$$

$$(\alpha, V) \rightarrow \alpha^\vee V.$$

3. Conservative vector fields on special Finsler manifolds

We have seen (c.f. Remark 6/(ii)) that for any function $\alpha \in C^\infty(M)$ the gradient vector field $V := \text{grad } \alpha^\vee$ is conservative. Then, as an easy calculation shows, $\text{div } V = 0$. In what follows we give some analogous results in case of special Finsler manifolds that it is essentially true vice-versa: if V is a conservative divergence-free vector field then it can be expressed (at least locally) as a linear combination of gradient vector fields with respect to the scalar multiplication introduced in Remark 6/(iii).

Theorem 2. *Let (M, E) be a two-dimensional, positive definite Finsler manifold and suppose that $V \in \mathfrak{X}^\vee(TM)$ is conservative. Then the following assertions are equivalent:*

- (i) $\text{div } V = 0$;
- (ii) for any chart $(U, (u^i)_{i=1}^2)$ on M ,

$$V = \alpha_1^\vee \text{grad } x^1 + \alpha_2^\vee \text{grad } x^2$$

$$(x^i := u^i \circ \pi, \alpha_i \in C^\infty(M), 1 \leq i \leq 2).$$

PROOF. First of all we recall that the divergence of a vertical vector field $JX \in \mathfrak{X}^\vee(TM)$ can be calculated by the formula

$$(29) \quad \text{div } JX = [\widetilde{J, JX}] + 2\tilde{\mathcal{C}}(X),$$

where $\widetilde{[J, JX]}$ and $\widetilde{\mathcal{C}}$ are the semibasic trace of the vector 1-form $[J, JX]$ and $\mathcal{C}$, respectively.

(For the definitions and proof see [4].)

Using Lemma 1 and (29) we get immediately the implication (ii) $\implies$ (i). It remains only to show that (i) $\implies$ (ii) is also valid. Let $(U, (u^i)_{i=1}^2)$ be an arbitrarily fixed chart on M . Then we have the following local expressions for the vector field V :

$$V \upharpoonright \pi_0^{-1}(U) = V^i \frac{\partial}{\partial y^i} = \beta_i \operatorname{grad} x^i,$$

where $V^i = g^{ij} \beta_j$ ($1 \leq i \leq 2$).

Formula (20) shows that

$$V^i g_{ij} = \frac{\partial}{\partial y^j} (VE)$$

and, consequently, $\beta_j = \frac{\partial}{\partial y^j} (VE)$ ($1 \leq j \leq 2$).

Using this coordinate expression of the functions β_1 and β_2 the homogeneity properties (21) give rise to the relations

$$(30) \quad C(\beta_j) = y^i \frac{\partial}{\partial y^i} (\beta_j) = 0 \quad (1 \leq j \leq 2);$$

$$(31) \quad y^i \frac{\partial}{\partial y^j} (\beta_i) = 0 \quad (1 \leq j \leq 2);$$

$$(32) \quad \frac{\partial \beta_i}{\partial y^j} = \frac{\partial \beta_j}{\partial y^i} \quad (1 \leq i, j \leq 2).$$

From the hypothesis $\operatorname{div} V = 0$ and (24), (26) we get that

$$\begin{aligned} 0 &= y^1 y^2 \left(g^{ij} \frac{\partial^2}{\partial y^i \partial y^j} (VE) \right) = y^1 y^2 \left(g^{ij} \frac{\partial}{\partial y^i} (\beta_j) \right) \\ &\stackrel{(32)}{=} y^1 y^2 \left(g^{11} \frac{\partial}{\partial y^1} (\beta_1) + 2g^{12} \frac{\partial}{\partial y^2} (\beta_1) + g^{22} \frac{\partial}{\partial y^2} (\beta_2) \right) \\ &\stackrel{(30), (31), (32)}{=} \left(- (y^2)^2 g^{11} + 2y^1 y^2 g^{12} - (y^1)^2 g^{22} \right) \frac{\partial}{\partial y^2} (\beta_1). \end{aligned}$$

Let $K \in \mathbb{R} \setminus \{0\}$ be an arbitrary *nonzero* real number and consider a tangent vector

$$v = v^1 \frac{\partial}{\partial u_p^1} + v^2 \frac{\partial}{\partial u_p^2} \in T_p M \setminus \{0\}$$

such that $v^2 = Kv^1$.

Suppose that

$$-(v^2)^2 g^{11}(v) + 2v^1 v^2 g^{12}(v) - (v^1)^2 g^{22}(v) = 0, \text{ i.e.}$$

$$\pm g^{12}(v) = \frac{g^{11}(v)|K| + g^{22}(v)\frac{1}{|K|}}{2} \geq \sqrt{g^{11}(v)g^{22}(v)} \Rightarrow \det g^{ij}(v) \leq 0,$$

which is a contradiction to the condition of positive definiteness. Thus we get that

$$\frac{\partial}{\partial y^2}(\beta_1) \stackrel{(32)}{=} \frac{\partial}{\partial y^1}(\beta_2) = 0$$

holds everywhere except at the points of a set of measure zero (in case of $v^1 = 0$ or $v^2 = 0$). Since the functions β_1 and β_2 are smooth over $\pi_0^{-1}(U)$, this means that for *any* tangent vector $v \in \pi_0^{-1}(U)$

$$(33) \quad \frac{\partial}{\partial y_v^2}(\beta_1) = \frac{\partial}{\partial y_v^1}(\beta_2) = 0.$$

Using the homogeneity properties (30) of the functions β_i ($1 \leq i \leq 2$) we can deduce the relations

$$(34) \quad \frac{\partial}{\partial y^1}(\beta_1) = \frac{\partial}{\partial y^2}(\beta_2) = 0$$

in a similar manner. (33) and (34) imply the functions β_1 and β_2 to be vertical lifts:

$$\beta_i = \alpha_i \circ \pi \quad (\alpha_i \in C^\infty(M), \ 1 \leq i \leq 2). \quad \square$$

Remark 7. (i) Since for any Finsler manifold the canonical spray S_h is divergence-free, i.e. $\operatorname{div} S_h = 0$, the conditions $\operatorname{div} V = 0$ and $\operatorname{div} \tilde{S} = 0$ ($\tilde{S} := S_h + V$) are equivalent. Indeed,

$$\begin{aligned} \mathcal{L}_{S_h} \omega^n &= \iota_{S_h} d\omega^n + d\iota_{S_h} \omega^n \stackrel{(F3)}{=} d\iota_{S_h} \omega^n \\ &= nd(\iota_{S_h} \omega \wedge \omega^{n-1}) \stackrel{(15), (F3)}{=} 0, \end{aligned}$$

so we get the vanishing of $\operatorname{div} S_h$.

(ii) It can be easily seen from the proof of Theorem 2 that in case of $\operatorname{div} V = 0$, the function $V(E)$ is necessarily linear on each fiber $T_p M$ ($p \in M$). Keeping our previous notations, for any chart $(U, (u^i)_{i=1}^2)$ on M ,

$$V(E) \upharpoonright \pi^{-1}(U) = y^1 \alpha_1^\vee + y^2 \alpha_2^\vee.$$

According to the implication (i) $\implies$ (ii) we can also say that the condition $\operatorname{div} V = 0$ determines a (locally) finitely generated nontrivial submodule of the module of conservative vector fields. (The trivial example of such a submodule is just the module $\mathfrak{X}(M)$; cf. Remark 6/(i), (iii).)

(iii) An alternative reasoning to prove the implication (i) $\implies$ (ii) can be realized as follows.

From the hypothesis $\operatorname{div} V = 0$ we get immediately that

$$\begin{aligned} 0 &\stackrel{(18)}{=} \mathcal{L}_V \omega^2 = \iota_V(d\omega^2) + d(\iota_V \omega^2) \stackrel{(\text{F3})}{=} d(\iota_V \omega^2) \\ &= d(2\iota_V \omega \wedge \omega) \stackrel{(20)}{=} 2(dd_J \beta) \wedge \omega, \end{aligned}$$

where $\beta := V(E)$.

Using the vanishing of the form $(dd_J \beta) \wedge \omega$ it follows that for any vector field $X \in \mathfrak{X}(M)$,

$$\iota_{X^\vee}[(dd_J \beta) \wedge \omega] = \iota_{X^h}[(dd_J \beta) \wedge \omega] = 0$$

and, consequently,

$$\begin{aligned} \iota_{X^\vee}(dd_J \beta) \wedge \omega &= -(dd_J \beta) \wedge \iota_{X^\vee} \omega, \\ \iota_{X^h}(dd_J \beta) \wedge \omega &= -(dd_J \beta) \wedge \iota_{X^h} \omega. \end{aligned}$$

By the help of these formulas it can be easily deduced that for any vector fields $X, Y \in \mathfrak{X}(M)$,

$$(dd_J \beta)(X^\vee, Y^h) \omega^2 = -2dd_J \beta \wedge \iota_{X^\vee} \omega \wedge \iota_{Y^h} \omega.$$

According to the tensorial character of our result we have:

$$(dd_J \beta)(JX, Y) \omega^2 = -2dd_J \beta \wedge \iota_{JX} \omega \wedge \iota_Y \omega \quad (X, Y \in \mathfrak{X}(TM)).$$

Let now $(U, (u^i)_{i=1}^2)$ be an arbitrary chart on M and, for brevity, let us set

$$\partial_k := \frac{\partial}{\partial y^k}, \quad \delta_k := \left(\frac{\partial}{\partial u^k} \right)^h, \quad \beta_k := \partial_k \beta \quad (1 \leq k \leq 2).$$

Then

$$\begin{aligned} dd_J \beta(J \operatorname{grad} \beta_k, \delta_k) \omega^2(\partial_1, \partial_2, \delta_1, \delta_2) \\ = -2dd_J \beta(J \operatorname{grad} \beta_k, \delta_k) \det [(g_{ij})_{1 \leq i, j \leq 2}]. \end{aligned}$$

On the other hand

$$\begin{aligned} dd_J \beta \wedge \iota_{J \operatorname{grad} \beta_k} \omega \wedge \iota_{\delta_k} \omega(\partial_1, \partial_2, \delta_1, \delta_2) &= g_{k1} \left((\partial_2 \partial_2 \beta)(\partial_1 \partial_k \beta) \right. \\ &\quad \left. - (\partial_1 \partial_2 \beta)(\partial_2 \partial_k \beta) \right) + g_{k2} \left((\partial_1 \partial_1 \beta)(\partial_2 \partial_k \beta) - (\partial_1 \partial_2 \beta)(\partial_1 \partial_k \beta) \right). \end{aligned}$$

Since

$$\begin{aligned} dd_J \beta(J \operatorname{grad} \beta_k, \delta_k) &= (J \operatorname{grad} \beta_k)(\beta_k) = \omega(\operatorname{grad} \beta_k, J \operatorname{grad} \beta_k) \\ &= -g(J \operatorname{grad} \beta_k, J \operatorname{grad} \beta_k) = -\|J \operatorname{grad} \beta_k\|^2, \end{aligned}$$

we get that

$$-\det [(g_{ij})_{1 \leq i, j \leq 2}] \cdot \sum_{k=1}^2 \|J \operatorname{grad} \beta_k\|^2 = \det [(\partial_i \partial_j \beta)_{1 \leq i, j \leq 2}] (g_{11} + g_{22}).$$

Suppose that

$$\det [(\partial_i \partial_j \beta)_{1 \leq i, j \leq 2}](v) \neq 0 \quad (v \in T_p M).$$

This means that the mapping

$$\begin{aligned} (\beta_1, \beta_2) : T_p M &\rightarrow \mathbb{R}^2 \\ w &\rightarrow (\beta_1, \beta_2)(w) := (\beta_1(w), \beta_2(w)) \end{aligned}$$

is a local diffeomorphism at the “point” $v \in T_p M$, which contradicts the homogeneity property of the functions β_1, β_2 . Indeed, these functions are obviously homogeneous of degree 0 (i.e. “constant along rays”); cf. Corollary 1.

The contradiction implies the vanishing of the vector fields $J \operatorname{grad} \beta_1$, $J \operatorname{grad} \beta_2$ and, consequently, it follows that β_1, β_2 are constant on each fiber $T_p M$ ($p \in M$). $\square$

Corollary 4. *Suppose that (M, E) is a two-dimensional, positive definite Finsler manifold. If $V \in \mathfrak{X}^v(TM)$ is a conservative vector field such that $\operatorname{div} V = 0$, then the induced horizontal endomorphism*

$$\tilde{h} := \frac{1}{2}(1 + [J, \tilde{S}]),$$

where $\tilde{S} := S_h + V$, has the following simple form:

$$\tilde{h} = h - \iota_{FV}\mathcal{C}.$$

(Cf. Remark 6/(ii).)

Definition. Let α be a Riemannian metric and β a (nonzero) 1-form on the manifold M . Consider the functions

$$(35) \quad \begin{cases} L_\alpha : TM \rightarrow \mathbb{R}, & v \rightarrow L_\alpha(v) := [\alpha_{\pi(v)}(v, v)]^{1/2}; \\ \tilde{\beta} : TM \rightarrow \mathbb{R}, & v \rightarrow \tilde{\beta}(v) := \beta_{\pi(v)}(v); \\ L := L_\alpha + \tilde{\beta}; & E := \frac{1}{2}L^2. \end{cases}$$

If

$$\|\tilde{\beta}\| := \sup_{v \in TM} \frac{\tilde{\beta}(v)}{L_\alpha(v)} < 1,$$

then (M, E) is a Finsler manifold which is said to be the *Randers manifold* constructed from the Riemann manifold (M, α) by the perturbation with $\tilde{\beta}$.

Remark 8. Consider a chart $(U, (u^i)_{i=1}^n)$ on M . If

$$\begin{aligned} \alpha_{ij} &:= \alpha \left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right), & (\alpha^{ij}) &:= (\alpha_{ij})^{-1}; \\ \beta_i &:= \beta \left(\frac{\partial}{\partial u^i} \right), & \beta^i &:= \alpha^{ij} \beta_j, \\ \beta^\# \upharpoonright U &= \beta^i \frac{\partial}{\partial u^i}, & b^2 &:= \|\beta^\#\|_\alpha^2 := \alpha(\beta^\#, \beta^\#) \end{aligned}$$

then we have the following coordinate expression (see e.g. [5], p. 209):

$$(36) \quad g^{ij} = \frac{L_\alpha}{L} (\alpha^{ij} \circ \pi) - \frac{L_\alpha}{L^2} (y^i (\beta^j \circ \pi) + y^j (\beta^i \circ \pi)) + \frac{L_\alpha (b^2 \circ \pi) + \tilde{\beta}}{L^3} y^i y^j.$$

Theorem 3. *Let (M, E) be a positive definite Randers manifold of dimension n and suppose that $V \in \mathfrak{X}^v(TM)$ is conservative. Then the following assertions are equivalent:*

- (i) $\operatorname{div} V = 0$;
- (ii) for any chart $(U, (u^i)_{i=1}^n)$ on M ,

$$V \upharpoonright \pi_0^{-1}(U) = \gamma_1^v \operatorname{grad} x^1 + \cdots + \gamma_n^v \operatorname{grad} x^n$$

$$(x^i := u^i \circ \pi, \gamma_i \in C^\infty(M), 1 \leq i \leq n).$$

PROOF. The implication (ii) $\implies$ (i) is trivial (cf. Theorem 2). To prove (i) $\implies$ (ii) first of all let

$$V \upharpoonright \pi_0^{-1}(U) = \delta_1 \operatorname{grad} x^1 + \cdots + \delta_n \operatorname{grad} x^n,$$

where $\delta_i \in C^\infty(\pi_0^{-1}(U))$, $1 \leq i \leq n$. In a similar manner as in the proof of Theorem 2, we get that for any indices $i \in \{1, \dots, n\}$

$$(37) \quad \delta_i = \frac{\partial}{\partial y^i} (VE).$$

(Note that the basic properties (30)–(32) of the coefficients are obviously not depend on the condition of dimensionality!)

The hypothesis $\operatorname{div} V = 0$ and (26) implies that

$$(38) \quad \begin{aligned} 0 &= \Delta(VE) \stackrel{(24)}{=} g^{ij} \frac{\partial^2}{\partial y^i \partial y^j} (VE) \stackrel{(36), (21)}{=} \\ &= \frac{L_\alpha}{L} (\alpha^{ij} \circ \pi) \frac{\partial^2}{\partial y^i \partial y^j} (VE) \Rightarrow \\ 0 &= (\alpha^{ij} \circ \pi) \frac{\partial^2}{\partial y^i \partial y^j} (VE) = \Delta_\alpha(VE), \end{aligned}$$

where Δ_α denotes the Brickell operator of the Finsler (especially Riemann) manifold (M, E_α) ; $E_\alpha := \frac{1}{2} L_\alpha^2$.

Differentiating (38) by $\frac{\partial}{\partial y^k}$ it follows that

$$0 = \frac{\partial}{\partial y^k} (\Delta_\alpha(VE)) \stackrel{(37)}{=} \Delta_\alpha \delta_k.$$

Thus Lemma 3 implies the functions $\delta_1, \dots, \delta_k$ to be vertical lifts, i.e.

$$\delta_k = \gamma_k \circ \pi \quad (\gamma_k \in C^\infty(M), \ 1 \leq k \leq n). \quad \square$$

Corollary 5. *Let (M, E) be a positive definite Randers manifold and suppose that $V \in \mathfrak{X}^v(TM)$ is a conservative vector field such that $\operatorname{div} V = 0$. Then the induced horizontal endomorphism*

$$\tilde{h} = \frac{1}{2}(1 + [J, \tilde{S}]),$$

where $\tilde{S} := S_h + V$, has the following simple form: $\tilde{h} = h - \iota_{FV}\mathcal{C}$.

(Cf. Corollary 4.)

Corollary 6. *Suppose that (M, E) is a (positive definite) Riemann manifold and let $V \in \mathfrak{X}^v(TM)$ be a conservative vector field.*

Then the following assertions are equivalent:

- (i) $\operatorname{div} V = 0$;
- (ii) $V = X^v$ ($X \in \mathfrak{X}(M)$), i.e. V is a vertical lift.

Moreover, the induced horizontal endomorphism $\tilde{h}$ coincides with the Barthel endomorphism.

PROOF. Since a Riemann manifold (M, E) can be considered as a special Randers manifold with $\tilde{\beta} \equiv 0$, Theorem 3 implies the equivalence

(iii) $\operatorname{div} V = 0$

(iv) $V \upharpoonright \pi_0^{-1}(U) = \gamma_1^v \operatorname{grad} x^1 + \dots + \gamma_n^v \operatorname{grad} x^n$ for an arbitrary chart $(U, (u^i)_{i=1}^n)$ on M .

Here, of course, all of vector fields $\operatorname{grad} x^i$ are vertical lifts, i.e.

$$\operatorname{grad} x^i = X_i^v \quad (X_i \in \mathfrak{X}(M), \ 1 \leq i \leq n).$$

(Note that $\operatorname{grad} x^i$ are just the vertical lifts of Riemannian gradients

$$\operatorname{grad}_R u^i \in \mathfrak{X}(U) \quad (1 \leq i \leq n).)$$

Consequently,

$$V \upharpoonright \pi_0^{-1}(U) = \gamma_1^\vee X_1^\vee + \cdots + \gamma_n^\vee X_n^\vee = (\gamma_1 X_1 + \cdots + \gamma_n X_n)^\vee.$$

The relation $\tilde{h} = h$ is trivial. □

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