

On zeros of reciprocal polynomials

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Abstract. The purpose of this paper is to show that all zeros of the reciprocal polynomial

$$P_m(z) = \sum_{k=0}^m A_k z^k \quad (z \in \mathbb{C})$$

of degree $m \geq 2$ with real coefficients $A_k \in \mathbb{R}$ (i.e. $A_m \neq 0$ and $A_k = A_{m-k}$ for all $k = 0, \dots, \lfloor \frac{m}{2} \rfloor$) are on the unit circle, provided that the “coefficient condition”

$$|A_m| \geq \sum_{k=1}^{m-1} |A_k - A_m|$$

is satisfied.

Moreover, if the “coefficient condition” holds, then all zeros e^{iu_j} , ($j = 1, 2, \dots, m$) can be arranged such that

$$\left| e^{i \frac{2\pi j}{m+1}} - e^{iu_j} \right| < \frac{\pi}{m+1} \quad (j = 1, \dots, m).$$

If $m = 2n + 1$ is odd, then $-1 = e^{iu_{n+1}}$ is always a zero, and all zeros of P_{2n+1} are single.

If $m = 2n$ is even, if the “coefficient condition” holds with equality and if

$$\operatorname{sgn} A_{2n} = \operatorname{sgn}(-1)^{k+1}(A_k - A_{2n}) = \operatorname{sgn}(-1)^{n+1} \frac{A_n - A_{2n}}{2} \quad (k = 1, 2, \dots, n-1),$$

then $u_n = u_{n+1} = \pi$, the number $-1 = e^{iu_n} = e^{iu_{n+1}}$ is a double zero of P_{2n} . Otherwise all zeros of P_{2n} are single.

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1. Introduction

The Coxeter transformation was introduced in the representation theory of finite dimensional algebras (see [2]). The characteristic polynomial of the Coxeter transformation of an oriented graph whose underlying graph is a wild star is a Salem polynomial (see [3], [4]).

Allowing circles in the underlying graph, the spectral properties of the Coxeter transformations get much more complicated. These properties are related to polynomials of the form

$$l(z^m + z^{m-1} + \cdots + z + 1) + (z^k + z^{m-k}) \quad (z \in \mathbb{C})$$

where m, k are fixed non-negative integers with $m \geq 2$, $1 \leq k \leq \lfloor \frac{m}{2} \rfloor$ and l is a fixed real number.

The zeros of the first expression $l(z^m + z^{m-1} + \cdots + z + 1)$ are

$$\epsilon_j = e^{i \frac{j}{m+1} 2\pi} \quad (j = 1, 2, \dots, m)$$

the $(m+1)$ st roots of unity except 1, *they are on the unit circle*. It is surprising that adding $z^k + z^{m-k}$ to the first expression the polynomial obtained inherits this property. Moreover, *not just all zeros remain on the unit circle but they move away from ϵ_j just a little* even if we add a linear combination $\sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} a_k(z^k + z^{m-k})$ to the expression $l(z^m + z^{m-1} + \cdots + z + 1)$, *provided that $|l|$ is large enough*. This leads to the main result of the paper: giving a sufficient condition for reciprocal polynomials to have all of their zeros on the unit circle and also giving the location of the zeros.

Our basic tool is a transformation of semi-reciprocal polynomials called the Chebyshev transformation. Although this transformation seems to be well known we could not find a suitable reference. In Section 2, based on [1], we summarize the properties of the Chebyshev transformation. In Section 3 we formulate our results and prove them. In Section 4 we discuss the necessity of our sufficient condition.

2. The Chebyshev transformation

A polynomial p of the form $p(z) = \sum_{j=0}^m a_j z^j$ ($z \in \mathbb{C}$) where $a_j \in \mathbb{C}$ are given numbers with $a_m \neq 0$, $a_j = a_{m-j}$ ($j = 0, \dots, \lfloor \frac{m}{2} \rfloor$) is called a *reciprocal polynomial of degree m* .

We need a more general class of reciprocal polynomials (of even degree).

Definition 1. A polynomial p of the form

$$(1) \quad p(z) = \sum_{j=0}^{2n} a_j z^j \quad (z \in \mathbb{C})$$

where $n \in \mathbb{N}$, $a_0, \dots, a_{2n} \in \mathbb{R}$ and

$$(2) \quad a_j = a_{2n-j} \quad (j = 0, \dots, n-1)$$

is called a *real semi-reciprocal polynomial* of degree at most $2n$. If $a_{2n} \neq 0$ we call p a *real reciprocal polynomial* of degree $2n$.

Denote by $\mathcal{R}_{2n}$ the set of all real semi-reciprocal polynomials of degree at most $2n$.

If $p \in \mathcal{R}_{2n}$, $p \neq o$ (o = the zero polynomial), then there is an integer k , $0 \leq k \leq n$, such that

$$(3) \quad \begin{aligned} a_{2n} = a_{2n-1} = \dots = a_{n+k+1} = 0 = a_{n-k-1} = \dots = a_0 \\ \text{but } a_{n+k} = a_{n-k} \neq 0. \end{aligned}$$

Hence

$$(4) \quad p(z) = \sum_{j=0}^{2n} a_j z^j = z^n \left[a_{n+k} \left(z^k + \frac{1}{z^k} \right) + \dots + a_{n+1} \left(z + \frac{1}{z} \right) + a_n \right].$$

Let T_j be the j th Chebyshev polynomial of the first kind, defined by

$$T_j(\cos x) = \cos jx \quad (j = 0, 1, \dots).$$

With $z + \frac{1}{z} = x$ we have $z^j + \frac{1}{z^j} = C_j(x)$ ($j = 1, 2, \dots$) (see e.g. [6], p. 224) where

$$C_j(x) := 2T_j\left(\frac{x}{2}\right) \quad (x \in \mathbb{C}, j = 1, 2, \dots)$$

are the normalized Chebyshev polynomials of the first kind. For us it will be now more convenient to define C_0 by

$$C_0(x) := T_0(x) \quad (x \in \mathbb{C}).$$

Hence, from (4)

$$(5) \quad p(z) = z^n \sum_{j=0}^k a_{n+j} C_j(x) = a_{n+k} z^n \prod_{j=1}^k (x - \alpha_j)$$

where $\alpha_j \in \mathbb{C}$ ($j = 1, \dots, k$) are the zeros of the polynomial $\sum_{j=0}^k a_{n+j} C_j(x)$.

Equation (5) remains true in the case when $k = 0$, i.e. $p(z) = a_n z^n$ if we agree that

$$(6) \quad \prod_{j=1}^0 b_j := 1.$$

Going back to the variable z we get that

$$p(z) = a_{n+k} z^{n-k} \prod_{j=1}^k z \left(z + \frac{1}{z} - \alpha_j \right) = a_{n+k} z^{n-k} \prod_{j=1}^k (z^2 - \alpha_j z + 1).$$

With this we have justified

Proposition 1. *Every non-zero polynomial $p \in \mathcal{R}_{2n}$ has the decomposition*

$$(7) \quad p(z) = a_{n+k} z^{n-k} \prod_{j=1}^k (z^2 - \alpha_j z + 1)$$

where $\alpha_1, \dots, \alpha_k \in \mathbb{C}$, $a_{n+k} \neq 0$ for some k with $0 \leq k \leq n$ and the convention (6) is adopted. If $p \in \mathcal{R}_{2n}$ is a reciprocal polynomial of degree $2n$, then (7) holds with $k = n$.

Definition 2. The Chebyshev transform of a non-zero polynomial $p \in \mathcal{R}_{2n}$ having the decomposition (7) is defined by

$$(8) \quad \mathcal{T}p(x) = a_{n+k} \prod_{j=1}^k (x - \alpha_j)$$

(with (6) adopted) while for the zero polynomial $p = o$ let

$$(9) \quad \mathcal{T}o(x) = 0.$$

It is clear that $\mathcal{T}$ maps $\mathcal{R}_{2n}$ into the set $\mathcal{P}_n$ of all polynomials of degree $\leq n$ with real coefficients.

Proposition 2. *The Chebyshev transform $\mathcal{T}$ is an isomorphism of the (real) vector space $\mathcal{R}_{2n}$ onto $\mathcal{P}_n$.*

PROOF. (i) $\mathcal{T}$ preserves the addition and the multiplication by a real constant. Using (5) and (3) (to include also the zero coefficients into the sum) we can write $\mathcal{T}p$ into the form

$$\mathcal{T}p(x) = a_{n+k} \prod_{j=1}^k (x - \alpha_j) = \sum_{j=0}^k a_{n+j} C_j(x) = \sum_{j=0}^n a_{n+j} C_j(x)$$

and the last form of $\mathcal{T}p$ is valid also for the zero polynomial. Taking now another $q \in \mathcal{R}_{2n}$ with $q(z) = \sum_{j=0}^{2n} b_j z^j$ ($b_j = b_{2n-j}$ for $j = 0, \dots, n-1$) and constants $\alpha, \beta \in \mathbb{R}$ we have

$$(\alpha p + \beta q)(z) = \sum_{j=0}^{2n} (\alpha a_j + \beta b_j) z^j$$

thus

$$\begin{aligned} \mathcal{T}(\alpha p + \beta q)(x) &= \sum_{j=0}^n (\alpha a_{n+j} + \beta b_{n+j}) C_j(x) \\ &= \alpha \sum_{j=0}^n a_{n+j} C_j(x) + \beta \sum_{j=0}^n b_{n+j} C_j(x) = \alpha (\mathcal{T}p(x)) + \beta (\mathcal{T}q(x)). \end{aligned}$$

(ii) $\mathcal{T}$ maps onto $\mathcal{P}_n$. Every polynomial $\tilde{r} \in \mathcal{P}_n$ can uniquely be written as a (real) linear combination of $C_0, C_1, \dots, C_n$ in the form $\tilde{r}(x) = \sum_{j=0}^n A_{n+j} C_j(x)$ ($A_{n+j} \in \mathbb{R}$). With $r(z) := \sum_{j=0}^{2n} A_j z^j$ where $A_j := A_{2n-j}$ for $j = 0, \dots, n-1$ we have $r \in \mathcal{R}_{2n}$ and $\mathcal{T}r = \tilde{r}$ proving our claim.

(iii) $\mathcal{T}$ is one-to-one. Namely, if $\mathcal{T}p = \mathcal{T}q$ for $p, q \in \mathcal{R}_{2n}$, then $\mathcal{T}p - \mathcal{T}q = \mathcal{T}(p - q) = o$ hence, by (8), (9) $p - q = o$, $p = q$. $\square$

Lemma 1. (i) *Let p be a real reciprocal polynomial of degree $2n$. Then all zeros of p are on the unit circle if and only if all zeros of its Chebyshev transform $\mathcal{T}p$ are in the closed interval $[-2, 2]$.*

(ii) *Moreover, if all zeros α_j of $\mathcal{T}p$ are in $[-2, 2]$, written as $\alpha_j = 2 \cos u_j$ with $u_j \in [0, \pi]$ ($j = 1, 2, \dots, n$), then all zeros of p are given by*

$$e^{\pm i u_j} \quad (j = 1, 2, \dots, n).$$

The multiplicity of $\alpha_j \neq \pm 2$ is the same as the multiplicities of e^{iu_j} and e^{-iu_j} ($j = 1, 2, \dots, n$) while in the case of $\alpha_j = \pm 2$ the multiplicities of the corresponding zeros $e^{iu_j} = \pm 1$ of p are doubled.

PROOF. (i) *Necessity.* Suppose that all zeros of p are on the unit circle. They can be arranged in conjugate pairs $(\beta_1, \bar{\beta}_1), (\beta_2, \bar{\beta}_2) \dots (\beta_n, \bar{\beta}_n)$. By assumption $|\beta_j|^2 = \beta_j \bar{\beta}_j = 1$, $\bar{\beta}_j = \frac{1}{\beta_j}$ ($j = 1, \dots, n$), hence

$$p(z) = a_{2n} \prod_{j=1}^n (z - \beta_j)(z - \bar{\beta}_j) = a_{2n} \prod_{j=1}^n (z^2 - (\beta_j + \bar{\beta}_j)z + 1)$$

and

$$\mathcal{T}p(x) = a_{2n} \prod_{j=1}^n (x - (\beta_j + \bar{\beta}_j)).$$

It is clear that $|\beta_j + \bar{\beta}_j| = |2\operatorname{Re}(\beta_j)| \leq 2|\beta_j| = 2$.

(i) *Sufficiency.* Assume that the Chebyshev transform has the form

$$\mathcal{T}p(x) = a_{2n} \prod_{j=1}^n (x - \alpha_j)$$

where $a_{2n} \neq 0$ and $\alpha_j \in [-2, 2]$ ($j = 1, \dots, n$). Then

$$p(z) = a_{2n} \prod_{j=1}^n (z^2 - \alpha_j z + 1).$$

Since $\alpha_j \in [-2, 2]$ we have $z^2 - \alpha_j z + 1 = (z - \beta_j)(z - \bar{\beta}_j)$ with $\beta_j \bar{\beta}_j = 1 = |\beta_j|^2$ proving that all zeros $\beta_1, \bar{\beta}_1, \beta_2, \bar{\beta}_2 \dots \beta_n, \bar{\beta}_n$ of p are on the unit circle.

(ii) We have $\alpha_j = 2 \cos u_j = \beta_j + \bar{\beta}_j$. Writing β_j as e^{iq_j} (here we may suppose that $0 \leq q_j \leq \pi$) we obtain that $2 \cos u_j = e^{iq_j} + e^{-iq_j} = 2 \cos q_j$ hence $u_j = q_j$ ($j = 1, 2, \dots, n$). The statement concerning the multiplicities is obvious. $\square$

3. Results and proofs

Theorem 1. *All zeros of the (real reciprocal) polynomial*

$$(10) \quad h_m(z) = l(z^m + z^{m-1} + \dots + z + 1) + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} a_k (z^{m-k} + z^k) \quad (z \in \mathbb{C})$$

of degree m where $l, a_1, \dots, a_{[\frac{m}{2}]} \in \mathbb{R}$, $l \neq 0$, $m \in \mathbb{N}$, $m \geq 2$, are on the unit circle if

$$(11) \quad |l| \geq 2 \sum_{k=1}^{[\frac{m}{2}]} |a_k|.$$

Moreover, if (11) is satisfied, then for even $m = 2n$ all zeros of h_m can be given as

$$e^{iu_j}, \quad e^{-iu_j} \quad (j = 1, 2, \dots, n)$$

where

$$\begin{aligned} \frac{j - \frac{1}{2}}{m+1} 2\pi < u_j < \frac{j + \frac{1}{2}}{m+1} 2\pi \quad (j = 1, 2, \dots, n-1) \\ \frac{n - \frac{1}{2}}{m+1} 2\pi < u_n \leq \pi. \end{aligned}$$

In the last inequality $u_n \leq \pi$, we have equality if and only if

$$(12) \quad |l| = 2 \sum_{k=1}^{[\frac{m}{2}]} |a_k| \quad \text{and} \quad \operatorname{sgn} l = \operatorname{sgn}(-1)^{k+1} \operatorname{sgn} a_k$$

for all $k = 1, 2, \dots, n$.

If (12) holds, then $-1 = e^{i\pi} = e^{-i\pi}$ is a double zero of h_m and all other zeros are single.

For odd $m = 2n + 1$ all zeros of h_m are single, they can be given as

$$-1, \quad e^{iu_j}, \quad e^{-iu_j} \quad (j = 1, 2, \dots, n)$$

where

$$\frac{j - \frac{1}{2}}{m+1} 2\pi < u_j < \frac{j + \frac{1}{2}}{m+1} 2\pi \quad (j = 1, 2, \dots, n).$$

Remark 1. The statement concerning the location of the zeros of h_m can also be formulated as follows.

If (11) is satisfied, then all the zeros e^{iu_j} ($j = 1, 2, \dots, m$) of h_m can be arranged such that

$$|\epsilon_j - e^{iu_j}| < \frac{\pi}{m+1} \quad (j = 1, \dots, m)$$

where, as in the introduction, ϵ_j are the $(m+1)$ st roots of unity, except 1.

Namely, for even $m = 2n$, let u_j ($j = 1, 2, \dots, n$) be the same as in Theorem 1 and $u_{n+j} := 2\pi - u_{n+1-j}$ ($j = 1, 2, \dots, n$). If (12) does not hold, then all zeros of h_m are single. If (12) holds, then $u_n = u_{n+1} = \pi$ and $-1 = e^{iu_n} = e^{iu_{n+1}}$ is a double zero and all other zeros are single.

For odd $m = 2n + 1$ let u_j ($j = 1, 2, \dots, n$) be the same as in Theorem 1, $u_{n+1} := \pi$ and $u_{n+1+j} := 2\pi - u_{n+1-j}$ ($j = 1, 2, \dots, n$). The number $-1 = e^{iu_{n+1}}$ is always a zero and all zeros are single.

PROOF. The basic idea of our proof is the following. Assume that (11) holds and let

$$x_j = 2 \cos \frac{j + \frac{1}{2}}{m+1} 2\pi \quad (j = 0, \dots, [\frac{m}{2}]).$$

If $m = 2n$ is an even number, we show that $\operatorname{sgn} \mathcal{T}h_{2n}(x_j) = \operatorname{sgn}(-1)^j \operatorname{sgn} l$ ($j = 0, 1, \dots, n-1$) and $\mathcal{T}h_{2n}(x_n) = 0$ if (12) holds, otherwise $\operatorname{sgn} \mathcal{T}h_{2n}(x_j) = \operatorname{sgn}(-1)^j \operatorname{sgn} l$ ($j = 0, \dots, n$).

If $m = 2n + 1$ is odd, then $h_{2n+1}(z) = (z+1)\bar{h}_{2n}(z)$ with a suitable reciprocal polynomial $\bar{h}_{2n}$ from $\mathcal{R}_{2n}$. We show that $\operatorname{sgn} \mathcal{T}\bar{h}_{2n}(x_j) = \operatorname{sgn} l \operatorname{sgn}(-1)^j$ ($j = 0, 1, \dots, n$).

Applying Lemma 1 completes the proof.

Case 1: $m = 2n$. With the notation $v_j(z) = z^j + z^{j-1} + \dots + 1 = \frac{z^{j+1}-1}{z-1}$, $e_j(z) = z^j$, $w_j(z) = z^j + 1$ ($j = 0, 1, \dots$) we have

$$h_{2n}(z) = lv_{2n}(z) + \sum_{k=1}^n a_k e_k(z) \cdot w_{2n-2k}(z),$$

$$\mathcal{T}h_{2n}(x) = l\mathcal{T}v_{2n}(x) + \sum_{k=1}^n a_k \mathcal{T}(e_k \cdot w_{2n-2k})(x).$$

The zeros of v_{2n} are the $(2n+1)$ st roots of unity, except 1: $e^{\frac{2j\pi i}{2n+1}}$ ($j = 1, 2, \dots, 2n$). They can be arranged into conjugate pairs: $(e^{\frac{2j\pi i}{2n+1}}, e^{\frac{2(2n+1-j)\pi i}{2n+1}}) = (e^{\frac{2j\pi i}{2n+1}}, e^{-\frac{2j\pi i}{2n+1}})$ ($j = 1, \dots, n$), thus

$$v_{2n}(z) = \prod_{j=1}^{2n} \left(z - e^{\frac{2j\pi i}{2n+1}} \right) = \prod_{j=1}^n \left(z - e^{\frac{2j\pi i}{2n+1}} \right) \left(z - e^{-\frac{2j\pi i}{2n+1}} \right)$$

$$= \prod_{j=1}^n \left(z^2 - 2 \cos \frac{2j\pi}{2n+1} z + 1 \right),$$

$$\mathcal{T}v_{2n}(x) = \prod_{j=1}^n \left(x - 2 \cos \frac{2j\pi}{2n+1} \right).$$

Similarly, for each $0 \leq k \leq n$ the zeros of w_{2n-2k} are the $(2n-2k)$ st roots of $-1 : e^{\frac{(2j-1)\pi i}{2n-2k}}$ ($j = 1, \dots, 2n-2k$). They can be arranged into conjugate pairs

$$\left(e^{\frac{(2j-1)\pi i}{2n-2k}}, e^{\frac{(2(2n-2k+1-j)-1)\pi i}{2n-2k}} \right) = \left(e^{\frac{(2j-1)\pi i}{2n-2k}}, e^{-\frac{(2j-1)\pi i}{2n-2k}} \right) \quad (j = 1, \dots, n-k).$$

Therefore

$$w_{2n-2k}(z) = \prod_{j=1}^{2n-2k} \left(z - e^{\frac{(2j-1)\pi i}{2n-2k}} \right) = \prod_{j=1}^{n-k} \left(z^2 - 2 \cos \frac{(2j-1)\pi}{2n-2k} z + 1 \right),$$

$$\mathcal{T}(e_k w_{2n-2k})(x) = \prod_{j=1}^{n-k} \left(x - 2 \cos \frac{(2j-1)\pi}{2n-2k} \right).$$

Denote by U_n the n th Chebyshev polynomial of the second kind (see for example in [6]), defined by

$$U_n(\cos x) = \frac{\sin(n+1)x}{\sin x} \quad (n = 0, 1, \dots).$$

We claim that

$$(13) \quad \mathcal{T}v_{2n}(x) = U_n\left(\frac{x}{2}\right) + U_{n-1}\left(\frac{x}{2}\right),$$

$$(14) \quad \mathcal{T}(e_k \cdot w_{2n-2k})(x) = 2U_{n-k}\left(\frac{x}{2}\right).$$

To justify the first identity we note that

$$(15) \quad \begin{aligned} U_n(\cos y) + U_{n-1}(\cos y) &= \frac{\sin(n+1)y + \sin ny}{\sin y} \\ &= 2 \frac{\sin \frac{(2n+1)y}{2} \cos \frac{y}{2}}{\sin y} = \frac{\sin \frac{(2n+1)y}{2}}{\sin \frac{y}{2}}. \end{aligned}$$

The right hand side is zero if and only if $y = \frac{2j\pi}{2n+1}$ ($j \in \mathbb{Z} \setminus \{0\}$) hence all zeros of $U_n\left(\frac{x}{2}\right) + U_{n-1}\left(\frac{x}{2}\right)$ are $2 \cos \frac{2j\pi}{2n+1}$ ($j = 1, \dots, n$). Since both sides of (13) are monics which have the same zeros, they are identical.

The zeros of T_p can be calculated easily from their definition, for $p \in \mathbb{N}$ they are

$$\cos \frac{(2j-1)\pi}{2p} \quad (j = 1, \dots, p).$$

Thus for $k < n$ the zeros of the monic $2T_{n-k}\left(\frac{x}{2}\right)$ are $2 \cos \frac{(2j-1)\pi}{2n-2k}$ ($j = 1, \dots, n-k$). They are the same as the zeros of $\mathcal{T}(e_k \cdot w_{2n-2k})$, hence (14) holds. It also holds for $k = n$ since then both sides of (14) are equal to 2.

Next we evaluate $\mathcal{Th}_{2n}$ at the points

$$x_j = 2 \cos \frac{j + \frac{1}{2}}{m+1} 2\pi \quad (j = 0, \dots, n)$$

of the interval $[-2, 2]$. Since $x_j = 2 \cos y_j$ with $y_j = \frac{j + \frac{1}{2}}{2n+1} 2\pi$ we have by (13), (14)

$$\begin{aligned} \mathcal{Th}_{2n}(x_j) &= l \left(U_n\left(\frac{x_j}{2}\right) + U_{n-1}\left(\frac{x_j}{2}\right) \right) + \sum_{k=1}^n 2a_k T_{n-k}\left(\frac{x_j}{2}\right) \\ &= 2 \left[\frac{\frac{l}{2} \sin \frac{2n+1}{2} y_j}{\sin \frac{1}{2} y_j} + \sum_{k=1}^n a_k \cos(n-k)y_j \right] \\ &= 2 \left[\frac{\frac{l}{2} (-1)^j}{\sin \frac{y_j}{2}} + \sum_{k=1}^n a_k \cos(n-k)y_j \right]. \end{aligned}$$

If $j = 0, 1, \dots, n-1$, then $0 < \sin \frac{y_j}{2} < 1$, $\sum_{k=1}^n |a_k \cos(n-k)y_j| \leq \sum_{k=1}^n |a_k|$ and by (11) the sign of the expression in the bracket is $(-1)^j \operatorname{sgn} l$.

If $j = n$, then $y_n = \pi$ and the expression in the bracket is

$$\frac{l}{2} (-1)^n + \sum_{k=1}^n a_k (-1)^{n-k} = (-1)^n \left(\frac{l}{2} + \sum_{k=1}^n a_k (-1)^k \right).$$

Its sign is $(-1)^n \operatorname{sgn} l$ if in (11) strict inequality holds or if in (11) we have equality and at least for one k ($1 \leq k \leq n$) we have $\operatorname{sgn} l = \operatorname{sgn}(-1)^k \operatorname{sgn} a_k$. If we have equality in (11) and $\operatorname{sgn} l = \operatorname{sgn}(-1)^{k+1} \operatorname{sgn} a_k$ for all $k = 1, \dots, n$, then the expression in the bracket is zero.

Thus either $\operatorname{sgn} \mathcal{T}h_{2n}(x_j) = \operatorname{sgn}(-1)^j \operatorname{sgn} l$ ($j = 0, \dots, n$) or $\operatorname{sgn} \mathcal{T}h_{2n}(x_j) = \operatorname{sgn}(-1)^j \operatorname{sgn} l$ ($j = 0, 1, \dots, n-1$) and $\mathcal{T}h_{2n}(x_n) = 0$. In both cases $\mathcal{T}h_{2n}$ has n distinct zeros in the interval $[-2, 2]$. Writing these in the form $2 \cos u_j$ with $0 \leq u_1 \leq u_2 \leq \dots \leq u_n \leq \pi$ and applying Lemma 1 we can complete the proof in the first case.

Case 2: $m = 2n + 1$. We have $h_{2n+1}(z) = (z + 1)\bar{h}_{2n}(z)$ with

$$\bar{h}_{2n}(z) = l\bar{v}_{2n}(z) + \sum_{k=1}^n a_k z^k \bar{w}_{2n-2k}(z)$$

where

$$\begin{aligned} \bar{v}_{2n}(z) &= z^{2n} + z^{2n-2} + \dots + z^2 + 1 = v_n(z^2), \\ \bar{w}_{2n-2k}(z) &= \frac{w_{2n+1-2k}(z)}{z+1} = \frac{z^{2n+1-2k} + 1}{z+1}. \end{aligned}$$

Using the factorization of v_n we get

$$\bar{v}_{2n}(z) = v_n(z^2) = \prod_{j=1}^n \left(z^2 - e^{\frac{2j\pi i}{n+1}} \right) = \prod_{j=1}^n \left(z - e^{\frac{j\pi i}{n+1}} \right) \left(z - e^{\frac{j\pi i}{n+1} - \pi i} \right).$$

Arranging the zeros of $\bar{v}_{2n}$ into conjugate pairs $(e^{\frac{j\pi i}{n+1}}, e^{-\frac{j\pi i}{n+1}})$ ($j = 1, \dots, n$) we have

$$\bar{v}_{2n}(z) = \prod_{j=1}^n \left(z - e^{\frac{j\pi i}{n+1}} \right) \left(z - e^{-\frac{j\pi i}{n+1}} \right) = \prod_{j=1}^n \left(z^2 - 2 \cos \frac{2j\pi}{2n+1} z + 1 \right)$$

therefore

$$\mathcal{T}\bar{v}_{2n}(x) = \prod_{j=1}^n \left(x - 2 \cos \frac{j\pi}{n+1} \right).$$

We can easily calculate the zeros of $\bar{w}_{2n-2k}$ (we omit this elementary calculation) and obtain the factorization

$$\begin{aligned} \bar{w}_{2n-2k}(z) &= \prod_{j=1}^{n-k} \left(z - e^{\frac{(2j-1)\pi i}{2n-2k+1}} \right) \left(z - e^{-\frac{(2j-1)\pi i}{2n-2k+1}} \right) \\ &= \prod_{j=1}^{n-k} \left(z^2 - 2 \cos \frac{(2j-1)\pi}{2n-2k+1} z + 1 \right) \end{aligned}$$

therefore

$$\mathcal{T}(e_k \cdot \bar{w}_{2n-2k})(x) = \prod_{j=1}^{n-k} \left(x - 2 \cos \frac{(2j-1)\pi}{2n-2k+1} \right).$$

Next we show that

$$(16) \quad \mathcal{T}\bar{v}_{2n}(x) = U_n \left(\frac{x}{2} \right),$$

$$(17) \quad \mathcal{T}(e_k \cdot \bar{w}_{2n-2k})(x) = U_{n-k} \left(\frac{x}{2} \right) - U_{n-k-1} \left(\frac{x}{2} \right).$$

where we have to adopt the convention

$$(18) \quad U_{-1}(x) = 0 \quad (x \in \mathbb{C}).$$

The first identity follows from the fact that the zeros of both sides are the same.

To justify the second we note that

$$\begin{aligned} U_{n-k}(\cos y) - U_{n-k-1}(\cos y) &= \frac{\sin(n-k+1)y - \sin(n-k)y}{\sin y} \\ &= \frac{2 \cos \frac{(2n-2k+1)y}{2} \sin \frac{y}{2}}{\sin y} = \frac{\cos \frac{(2n-2k+1)y}{2}}{\cos \frac{y}{2}} \end{aligned}$$

for all $k = 0, \dots, n$ provided that the convention (18) is adopted.

If $k = n$, then both sides of (17) are equal to 1 thus (17) holds. For $k < n$ the right hand side of (17) is zero if and only if $y = \frac{(2j-1)\pi}{2n-2k+1}$ ($j \in \mathbb{Z}$) hence all zeros of $U_{n-k} \left(\frac{x}{2} \right) - U_{n-k-1} \left(\frac{x}{2} \right)$ are $2 \cos \frac{(2j-1)\pi}{2n-2k+1}$ ($j = 1, \dots, n-k$), they are the same as the zeros of $\mathcal{T}(e_k \cdot \bar{w}_{2n-2k})$ proving (17).

By the linearity of the Chebyshev transform and by (16), (17) we have

$$\begin{aligned} \mathcal{T}\bar{h}_{2n}(x) &= l\mathcal{T}\bar{v}_{2n}(x) + \sum_{k=1}^n a_k \mathcal{T}(e_k \cdot \bar{w}_{2n-2k})(x) \\ &= lU_n \left(\frac{x}{2} \right) + \sum_{k=1}^n a_k \left[U_{n-k} \left(\frac{x}{2} \right) - U_{n-k-1} \left(\frac{x}{2} \right) \right]. \end{aligned}$$

Next we evaluate $\mathcal{T}\bar{h}_{2n}$ at the points

$$\bar{x}_j = x_j = 2 \cos \frac{j + \frac{1}{2}}{2n+2} 2\pi \quad (j = 0, \dots, n)$$

of the interval $[-2, 2]$. Since $\bar{x}_j = 2 \cos \bar{y}_j$ with $\bar{y}_j = \frac{j + \frac{1}{2}}{2n+2} 2\pi$ we have

$$\begin{aligned} \mathcal{T}\bar{h}_{2n}(\bar{x}_j) &= 2 \left[\frac{l \sin(n+1)\bar{y}_j}{2 \sin \bar{y}_j} + \frac{\sum_{k=1}^n a_k \cos \frac{2n-2k+1}{2} \bar{y}_j}{2 \cos \frac{\bar{y}_j}{2}} \right] \\ &= 2 \left[\frac{l (-1)^j}{2 \sin \bar{y}_j} + \sum_{k=1}^n a_k \frac{\cos \frac{2n-2k+1}{2} \bar{y}_j}{2 \cos \frac{\bar{y}_j}{2}} \right] \\ &= 2 \frac{\frac{l}{2} (-1)^j + \sum_{k=1}^n a_k \sin \frac{\bar{y}_j}{2} \cos \frac{2n-2k+1}{2} \bar{y}_j}{\sin \bar{y}_j}. \end{aligned}$$

Since $\bar{y}_j \in]0, \pi[$ we have $\sin \bar{y}_j > 0$, $0 < \sin \frac{\bar{y}_j}{2} < 1$, $|\cos \frac{2n-2k+1}{2} \bar{y}_j| \leq 1$ for all $k = 1, \dots, n$ therefore the sign of the expression in the bracket is $\text{sgn } l \text{sgn}(-1)^j$. Thus $\text{sgn}(\mathcal{T}\bar{h}_{2n}(x_j)) = \text{sgn } l \text{sgn}(-1)^j$ ($j = 0, 1, \dots, n$) proving that $\mathcal{T}\bar{h}_{2n}$ has n different zeros in $[-2, 2]$. Writing these zeros in the form $2 \cos u_j$ with $0 \leq u_1 \leq u_2 \leq \dots \leq u_n \leq \pi$ and applying Lemma 1 the proof is completed in the second case as well. $\square$

We can formulate Theorem 1 in a more symmetric way. This formulation explains, in a certain way, the appearance of the factor 2 in (11).

Theorem 2. *All zeros of the reciprocal polynomial*

$$(19) \quad P_m(z) = \sum_{k=0}^m A_k z^k \quad (z \in \mathbb{C})$$

of degree $m \geq 2$ with real coefficients $A_k \in \mathbb{R}$ (i.e. $A_m \neq 0$ and $A_k = A_{m-k}$ for all $k = 0, \dots, [\frac{m}{2}]$) are on the unit circle, provided that

$$(20) \quad |A_m| \geq \sum_{k=1}^{m-1} |A_k - A_m|.$$

If (20) holds, then all zeros e^{iu_j} ($j = 1, 2, \dots, m$) of P_m can be arranged such that

$$|\epsilon_j - e^{iu_j}| < \frac{\pi}{m+1} \quad (j = 1, \dots, m).$$

If $m = 2n + 1$ is odd, then $-1 = e^{iu_{n+1}}$ is always a zero and all zeros of P_m are single.

If $m = 2n$ is even

$$(21) \quad \begin{cases} |A_{2n}| = \sum_{k=1}^{2n-1} |A_k - A_{2n}| & \text{and} \\ \operatorname{sgn} A_{2n} = \operatorname{sgn}(-1)^{k+1}(A_k - A_{2n}) = \operatorname{sgn}(-1)^{n+1} \frac{A_n - A_{2n}}{2} \\ (k = 1, 2, \dots, n-1) \end{cases}$$

holds, then $u_n = u_{n+1} = \pi$, the number $-1 = e^{iu_n} = e^{iu_{n+1}}$ is a double zero of P_m and all other zeros are single. Otherwise (i.e. if $m = 2n$, (21) does not hold) all zeros of P_m are single.

PROOF. Comparing the coefficients of z^j in h_m and P_m we see that for even $m = 2n$

$$A_{2n} = A_0 = l, \quad A_{2n-1} = A_1 = l + a_1, \dots, \quad A_{n+1} = A_{n-1} = l + a_{n-1},$$

$$A_n = l + 2a_n$$

thus $l = A_{2n}$, $a_k = A_{2n-k} - A_{2n} = A_k - A_{2n}$ for $k = 1, 2, \dots, n-1$ and $2a_n = A_n - A_{2n}$. Therefore the condition (11)

$$|l| \geq 2 \sum_{k=1}^n |a_k|$$

can be written as

$$|A_{2n}| \geq 2 \sum_{k=1}^{n-1} |A_k - A_{2n}| + |A_n - A_{2n}| = \sum_{k=1}^{2n-1} |A_k - A_{2n}|$$

which is the same as (20).

For odd $m = 2n + 1$ the comparison of the coefficients gives that

$$A_{2n+1} = A_0 = l, \quad A_{2n} = A_1 = l + a_1, \dots, \quad A_{n+1} = A_n = l + a_n$$

thus $l = A_{2n+1}$, $a_k = A_{2n+1-k} - A_{2n+1} = A_k - A_{2n+1}$ for $k = 1, 2, \dots, n$

and (11) can be written as

$$\begin{aligned} |A_{2n+1}| &\geq 2 \sum_{k=1}^n |A_k - A_{2n+1}| = \sum_{k=1}^n (|A_k - A_{2n+1}| + |A_{2n+1-k} - A_{2n+1}|) \\ &= \sum_{k=1}^{2n} |A_k - A_{2n+1}| \end{aligned}$$

proving (20). The statement concerning the location of the zeros follows from Remark 1. $\square$

4. Necessary and sufficient conditions

If the degree m of P_m is small we can easily obtain necessary and sufficient conditions for all zeros of P_m to be on the unit circle.

If $m = 2$, then $P_2(z) = A_2 z^2 + A_1 z + A_2 = z(A_2(z + \frac{1}{z}) + A_1)$ hence $\mathcal{T}P_2(x) = A_2 x + A_1$. The only zero of $\mathcal{T}P_2$ is in $[-2, 2]$ if and only if

$$(22) \quad |A_2| \geq \frac{1}{2}|A_1|.$$

This is the criteria for P_2 to have all zeros on the unit circle.

If $m = 3$, then $P_3(z) = A_3 z^3 + A_2 z^2 + A_2 z + A_3 = (z + 1)(A_3 z^2 + (A_2 - A_3)z + A_3)$. By (22) the zeros of P_3 are on the unit circle if and only if

$$(23) \quad |A_3| \geq \frac{1}{2}|A_2 - A_3|.$$

If $m = 4$, then $P_4(z) = A_4 z^4 + A_3 z^3 + A_2 z^2 + A_3 z + A_4 = z^2(A_4(z^2 + \frac{1}{z^2}) + A_3(z + \frac{1}{z}) + A_2)$ hence with $x = z + \frac{1}{z}$ we get that $\mathcal{T}P_4(x) = A_4(x^2 - 2) + A_3 x + A_2$. By Lemma 1 all zeros of P_4 are on the unit circle if and only if the discriminant of $\mathcal{T}P_4$ is non-negative:

$$(24) \quad A_3^2 - 4A_4(A_2 - 2A_4) \geq 0$$

and

$$(25) \quad -2 \leq x_1, \quad x_2 \leq 2$$

hold where $x_1 \leq x_2$ are the real zeros of $\mathcal{T}P_4$. A simple calculation shows that (24) and (25) are equivalent to

$$(26) \quad \begin{aligned} 2\sqrt{\max\{A_2A_4 - 2A_4^2, 0\}} &\leq |A_3| \\ &\leq \min\left\{4|A_4|, |A_4| + \frac{1}{2}A_2 \operatorname{sgn} A_4\right\}. \end{aligned}$$

This is the criterion for P_4 to have all of its zeros on the unit circle.

For $m = 2$ (22) holds if and only if

$$A_1 \in [-2|A_2|, 2|A_2|]$$

while (20) gives only the smaller interval

$$A_1 \in [A_2 - |A_2|, A_2 + |A_2|].$$

This shows that (20) for $m = 2$ is not necessary. The situation is similar for $m = 3$.

For $m = 4$ the necessary and sufficient condition (26) is non-linear in the coefficients, while our sufficient condition (20) is linear for all $m \geq 2$. In some special cases we get necessary and sufficient conditions.

Corollary 1. *All zeros of the polynomial*

$$l(z^m + z^{m-1} + \cdots + z + 1) + (z^k + z^{m-k}) \quad (z \in \mathbb{C})$$

where m, k are fixed non-negative integers with $m \geq 2$, $1 \leq k \leq \lfloor \frac{m}{2} \rfloor$ and l is a fixed positive number, are on the unit circle for all $m \geq 2$ and for all $k = 1, 2, \dots, \lfloor \frac{m}{2} \rfloor$ if and only if $l \geq 2$.

Namely, taking $m = 2$, $k = 1$ by (22) all zeros of the resulting polynomial $lz^2 + (l+2)z + l$ are on the unit circle if and only if $l \notin (-\frac{2}{3}, 2)$ therefore $l \geq 2$. On the other hand if $l \geq 2$, then by Theorem 1 all zeros of the polynomial $l(z^m + z^{m-1} + \cdots + z + 1) + (z^k + z^{m-k})$ are on the unit circle.

Remark 2. A preliminary version of some parts of this paper was reported in [5].

A. SCHINZEL [7] generalized Theorem 2 to the case of self-inversive polynomials over $\mathbb{C}$, i.e. polynomials $P_m(z) = \sum_{k=0}^n A_k z^k$ for which $A_k \in \mathbb{C}$,

$A_m \neq 0$, $\epsilon \bar{A}_k = A_{m-k}$ for all $k = 0, \dots, m$ with a fixed $\epsilon \in \mathbb{C}$, $|\epsilon| = 1$. He proved that all zeros of P_m are on the unit circle, provided that

$$|A_m| \geq \inf \sum_{k=0}^m |cA_k - d^{m-k}A_m|,$$

where the infimum is taken over all $c, d \in \mathbb{C}$ and $|d| = 1$.

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