

Jensen's equation and bisymmetry

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Abstract. In this note we give short proofs for some known results on the Jensen equation and characterizations of quasi-linear means by the bisymmetry, and of quasi-arithmetic means as given by KOLMOGOROV [6], NAGUMO [9] and DE FINETTI [5].

1. Introduction

Let $I \subset \mathbb{R}$ (the reals) be an interval of positive length, n be a fixed positive integer and $\alpha \in]0, 1[$ be fixed. The Jensen equation is

$$(1) \quad f(\alpha u + (1 - \alpha)v) = \alpha f(u) + (1 - \alpha)f(v)$$

where $f : I^n \rightarrow \mathbb{R}$ and (1) holds for all $u, v \in I^n$. A function $B : I^n \rightarrow I$ is n -bisymmetric if $n \geq 2$ (fixed) and

$$(2) \quad B(B(x_{11}, \dots, x_{1n}), \dots, B(x_{n1}, \dots, x_{nn})) \\ = B(B(x_{11}, \dots, x_{n1}), \dots, B(x_{1n}, \dots, x_{nn}))$$

Mathematics Subject Classification: 39B22.

Key words and phrases: Jensen's equation, bisymmetry, quasi-linear means, quasi-arithmetic means.

This research has been supported by OTKA, Grant T-030082 and by FKFP, Grant 0215/2001.

holds for all $x_{ij} \in I$ ($i, j = 1, \dots, n$). In what follows we consider equation (2) as application of the function $B : I^n \rightarrow I$ to the matrix

$$\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & & & \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{pmatrix}.$$

An element $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ is called an n -weight if $\lambda_k \in]0, 1[$ ($k = 1, \dots, n$) and $\sum_{k=1}^n \lambda_k = 1$. Throughout the paper, for $n \geq 2$, $\text{CSIR}(I^n)$ denotes the class of all continuous functions $g : I^n \rightarrow \mathbb{R}$ that are strictly increasing in each variable and n -reflexive, that is,

$$g(x, \dots, x) = x \quad \text{for all } x \in I.$$

In this note we are interested in solutions of (1) and (2) belonging to the class $\text{CSIR}(I^n)$. These solutions are known (see MÜNNICH–MAKSA–MOKKEN [8] and its references). Our aim is to simplify the proofs by pointing out the important role played by the Jensen equation (1) in characterizing quasi-linear means by bisymmetry and quasi-arithmetic means by the axioms of KOLMOGOROV [6], NAGUMO [9] and DE FINETTI [5].

2. Solutions of the Jensen equation belonging to $\text{CSIR}(I^n)$

These solutions can be obtained from the case $n = 2$ (see ACZÉL [1]) by induction on n as in [8] or from more general results (see e.g. KUCZMA [7] for $\alpha = \frac{1}{2}$ and DARÓCZY–MAKSA [4] for $\alpha \neq \frac{1}{2}$). Here we give an elementary proof, without induction, of the following.

Lemma. *Suppose that $f \in \text{CSIR}(I^n)$ is a solution of equation (1) for an $\alpha \in]0, 1[$. Then*

$$(3) \quad f(u) = \sum_{k=1}^n \lambda_k u_k$$

for all $u = (u_1, \dots, u_n) \in I^n$ and for an n -weight $(\lambda_1, \dots, \lambda_n)$.

PROOF. Let $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n) \in I^n$, $a, b \in I$, $a < b$ and $k \in \{1, \dots, n\}$ be fixed. Integrate both sides of (1) on $[a, b]$ with respect to v_k .

Then we have that

$$\begin{aligned} & \frac{1}{1-\alpha} \int_{\alpha u_k + (1-\alpha)a}^{\alpha u_k + (1-\alpha)b} f(\alpha u_1 + (1-\alpha)v_1, \dots, \overset{k}{t}, \dots, \alpha u_n + (1-\alpha)v_n) dt \\ &= \alpha(b-a)f(u_1, \dots, u_k, \dots, u_n) + (1-\alpha) \int_a^b f(v_1, \dots, v_k, \dots, v_n) dv_k, \end{aligned}$$

whence we get the existence and continuity of the partial derivative function $\partial_k f$. Therefore f is continuously differentiable. Differentiating both sides of (1) with respect to u_k and with respect to v_k we obtain that $\partial_k f(\alpha u + (1-\alpha)v) = \partial_k f(u)$ and $\partial_k f(\alpha u + (1-\alpha)v) = \partial_k f(v)$, respectively. Thus $f'(u) = f'(v)$, that is, f' is constant. Therefore there exists $(\lambda_0, \lambda_1, \dots, \lambda_n) \in \mathbb{R}^{n+1}$ such that

$$f(u) = \sum_{k=1}^n \lambda_k u_k + \lambda_0, \quad (u = (u_1, \dots, u_n) \in I^n).$$

Since f is reflexive and strictly monotone increasing in each variable we have that $\lambda_0 = 0$ and $(\lambda_1, \dots, \lambda_n)$ is an n -weight. $\square$

3. n -bisymmetric elements of $\text{CSIR}(I^n)$

The basic tool for finding all n -bisymmetric functions belonging to $\text{CSIR}(I^n)$ is the classical theorem of J. ACZÉL for the case $n = 2$ (see [1], [2], [3]). The extension of this result to $n > 2$ in the not necessarily symmetric case was first proven in [8]. Here we present a much simpler proof than in [8].

Theorem. *A function $B \in \text{CSIR}(I^n)$ is a solution of (2), if and only if, there exist an n -weight $(\lambda_1, \dots, \lambda_n)$ and a continuous and strictly increasing function $\varphi : I \rightarrow \mathbb{R}$ such that*

$$(4) \quad B(x_1, \dots, x_n) = \varphi^{-1} \left(\sum_{k=1}^n \lambda_k \varphi(x_k) \right) \quad (x_1, \dots, x_n \in I).$$

PROOF. The “if” part is obvious. The “only if” part is true for $n = 2$ (see e.g. [1]). Let $x, y, u, v \in I$ and apply B to the $n \times n$ matrix

$$\begin{pmatrix} x & y & \dots & y \\ u & v & \dots & v \\ \vdots & \vdots & & \vdots \\ u & v & \dots & v \end{pmatrix}.$$

Then, with the definition

$$(5) \quad M(x, y) = B(x, y, \dots, y), \quad (x, y \in I)$$

(2) implies that

$$M(M(x, y), M(u, v)) = M(M(x, u), M(y, v)).$$

Since $M \in \text{CSIR}(I^2)$ we can apply ACZÉL’s theorem (see [1], [2], [3]) to get

$$(6) \quad M(x, y) = \varphi^{-1}(\alpha\varphi(x) + (1 - \alpha)\varphi(y)) \quad (x, y \in I)$$

for a $\varphi : I \rightarrow \mathbb{R}$ strictly increasing and continuous function and for an $\alpha \in]0, 1[$. Let now $(x_1, \dots, x_n), (y_1, \dots, y_n) \in I^n$ and apply B to the $n \times n$ matrix

$$\begin{pmatrix} x_1 & y_1 & \dots & y_1 \\ x_2 & y_2 & \dots & y_2 \\ \vdots & \vdots & & \vdots \\ x_n & y_n & \dots & y_n \end{pmatrix}.$$

Then, by (5) and (6), we obtain that

$$(7) \quad B(\varphi^{-1}(\alpha\varphi(x_1) + (1 - \alpha)\varphi(y_1)), \dots, \varphi^{-1}(\alpha\varphi(x_n) + (1 - \alpha)\varphi(y_n))) \\ = \varphi^{-1}(\alpha\varphi(B(x_1, \dots, x_n)) + (1 - \alpha)\varphi(B(y_1, \dots, y_n))).$$

Let $J = \varphi(I)$. Then $J \subset \mathbb{R}$ is an interval of positive length, and for all $u_1, \dots, u_n \in J$, with the substitutions $x_k = \varphi^{-1}(u_k)$ ($k = 1, \dots, n$) and with the definition

$$(8) \quad f(u_1, \dots, u_n) = \varphi(B(\varphi^{-1}(u_1), \dots, \varphi^{-1}(u_n)))$$

equation (7) goes over into Jensen equation (1). Applying our lemma we have (3) with some n -weight $(\lambda_1, \dots, \lambda_n)$. Thus (8) and (3) imply (4). $\square$

4. Remarks on the characterizations of quasi-linear and quasi-arithmetic means

In the sense of KOLMOGOROV [6], NAGUMO [9], and de FINETTI [5] a quasi-arithmetic mean is a sequence (B_n) of functions $B_n : I^n \rightarrow I$ ($n \geq 2$) with the property that there exists a strictly increasing and continuous function $\varphi : I \rightarrow \mathbb{R}$ such that

$$(9) \quad B_n(x_1, \dots, x_n) = \varphi^{-1} \left(\frac{1}{n} \sum_{k=1}^n \varphi(x_k) \right)$$

for all positive integers $n \geq 2$ and for all $(x_1, \dots, x_n) \in I^n$. For a characterization of this sequence (B_n) they used the following system of axioms:

- (a) $B_n \in \text{CSIR}(I^n)$ for all $n \geq 2$;
- (b) B_n is symmetric for all $n \geq 2$;
- (c) $B_n(B_k(x_1, \dots, x_k), \dots, B_k(x_1, \dots, x_k), x_{k+1}, \dots, x_n) = B_n(x_1, \dots, x_n)$ for all $n \geq 2$, for all $2 \leq k \leq n$ and for all $x_1, \dots, x_n \in I$.

ACZÉL [3] has observed that (b) and (c) imply that B_n is n -bisymmetric for all fixed $n \geq 2$ while the converse is not true. Indeed, let $x_{ij} \in I$ and $y_i = B_n(x_{i1}, \dots, x_{in})$ ($i, j = 1, \dots, n$). Then (b), (c) and the reflexivity imply that

$$\begin{aligned} & B_{n^2}(x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, \dots, x_{n1}, \dots, x_{nn}) \\ &= B_{n^2}(y_1, \dots, y_1, y_2, \dots, y_2, \dots, y_n, \dots, y_n) \\ &= B_{n^2}(y_1, y_2, \dots, y_n, y_1, y_2, \dots, y_n, \dots, y_1, y_2, \dots, y_n) \\ &= B_{n^2}(B_n(y_1, \dots, y_n), B_n(y_1, \dots, y_n), \dots, B_n(y_1, \dots, y_n)) \\ &= B_n(y_1, \dots, y_n) \\ &= B_n(B_n(x_{11}, \dots, x_{1n}), B_n(x_{21}, \dots, x_{2n}), \dots, B_n(x_{n1}, \dots, x_{nn})). \end{aligned}$$

This, again by the symmetry of B_{n^2} , implies (2) for B_n instead of B . On the other hand, an easy calculation shows that, (2) is satisfied by any function $B : I^n \rightarrow I$ of the form

$$(10) \quad B(x_1, \dots, x_n) = \varphi^{-1} \left(\sum_{k=1}^n \lambda_k \varphi(x_k) \right), \quad ((x_1, \dots, x_n) \in I^n)$$

with strictly increasing and continuous $\varphi: I \rightarrow \mathbb{R}$ and n -weight $(\lambda_1, \dots, \lambda_n)$. A function B defined by (10) is called quasi-linear mean of n variables with n -weight $(\lambda_1, \dots, \lambda_n)$ (see [3]) and it is symmetric, if $\lambda_1 = \dots = \lambda_n = \frac{1}{n}$. Therefore our theorem characterizes the quasi-linear means of n variables for fixed $n \geq 2$. Furthermore this theorem and our lemma provide the following short proof of the characterization based on the properties (a)–(c) of a quasi-arithmetic mean (B_n) : It is obvious that any quasi-arithmetic mean (B_n) has the properties (a)–(c). To prove the converse suppose that (B_n) satisfies (a)–(c). Then $B_n \in \text{CSIR}(I^n)$ is a symmetric solution of (2) for all fixed $n \geq 2$. Therefore, by our theorem, there exists a strictly monotone increasing and continuous function $\varphi_n: I \rightarrow \mathbb{R}$ such that

$$(11) \quad B_n(x_1, \dots, x_n) = \varphi_n^{-1} \left(\frac{1}{n} \sum_{k=1}^n \varphi_n(x_k) \right), \quad ((x_1, \dots, x_n) \in I^n).$$

Write $\varphi = \varphi_2$ and use equation (c) for $k = 2$. Then (11) implies that

$$(12) \quad \varphi_n^{-1} \left(\frac{\varphi_n(x_1) + \varphi_n(x_2)}{2} \right) = \varphi^{-1} \left(\frac{\varphi(x_1) + \varphi(x_2)}{2} \right)$$

if $x_1, x_2 \in I$.

It is well-known (see e.g. [1]) and easy to see that (12) can be reduced to the one-dimensional Jensen equation for $f = \varphi_n \circ \varphi^{-1}$ (with $\alpha = \frac{1}{2}$) and we have that $\varphi_n(x) = a_n \varphi(x) + b_n$ for all $x \in I$ and for some $0 < a_n \in \mathbb{R}$, $b_n \in \mathbb{R}$. Finally, this and (11) imply (9), that is, (B_n) is a quasi-arithmetic mean.

Acknowledgement. The author is grateful to ZSOLT PÁLES for the idea by which the induction argument in the first version of the proof of the Theorem became unnecessary.

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(Received April 18, 2002; revised September 12, 2002)