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On the number of simple zeros of certain polynomials

By ÁKOS PINTÉR (Debrecen)

Dedicated to Professor Lajos Tamássy on his 70th birthday

1. Introduction

The purpose of the present note is to study the number of simple zeros of a large class of integer-valued polynomials. Combining our result with a well-known theorem on superelliptic equations we obtain an effective finiteness statement for a general diophantine equation. The proof of our Theorem is based on some properties of the canonical mapping $\mathbb{Z}[X] \to \mathbb{Z}_p[X]$ (p is a prime). We remark that this method has been applied fruitfully by several authors, especially to Bernoulli polynomials; see e.g. [2],[3],[4],[5],[9],[10].

2. The Theorem

Let n be a positive integer, and set $\binom{X}{n} = \frac{X(X-1)\cdots(X-n+1)}{n!}$. Furthermore, let $f(X)$ be an integer-valued polynomial with $\deg f(X) \leq n-1$, and let $g(X) \in \mathbb{Z}[X]$.

Theorem. *Suppose that $n \geq 6$ and let p denote a prime for which*

$$\frac{2}{3}n < p \leq n.$$

If a_n is an integer not divisible by p then the polynomial

$$F(X) = a_n \binom{X}{n} + f(X) + g(X)$$

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has at least $\left[\frac{n}{3}\right]+1$ *simple zeros.*

PROOF. Put $f_i(X)=X(X-1)\cdots(X-i+1)$ for $i=1,\dots,n$ and $f_0(X)=1$. Since $f(X)$ is an integer-valued polynomial, thus (cf. [7])

$$f(X)=a_{n-1}\binom{X}{n-1}+\ldots+a_1\binom{X}{1}+a_0,$$

where the coefficients $a_{n-1},\dots,a_1,a_0$ are rational integers. We get

$$n!F(X)=a_nf_n(X)+\ldots+a_pn(n-1)\cdots(p+1)f_p(X)+\ldots+n!a_0+n!g(X)\in\mathbb{Z}[X]\,.$$

For $S(X)\in\mathbb{Z}[X]$, we denote by $(S(X))_p$ the image of S in $\mathbb{Z}_p[X]$ under the canonical homomorphism $\mathbb{Z}\to\mathbb{Z}_p$. There is a $h(X)\in\mathbb{Z}[X]$ such that

$$(n!F(X))_p=(f_p(X))_p(h(X))_p$$

and $\deg(h(X))_p=n-p$. Since all the zeros of $(f_p(X))_p$ are simple, the polynomial $(n!F(X))_p$ as well as the polynomial $n!F(X)$ has at least $p-(n-p)=2p-n>\frac{n}{3}$ simple zeros.

3. An application to diophantine equations

Let $F(X)$ be as above, and let a be a non-zero integer.

Corollary. *All the solutions of the equation*

$$F(x)=ay^m \qquad \text{in integers } x,y,m \text{ with } |y|>1 \text{ and } m>1 \tag{1}$$

satisfy

$$\max(x,|y|,m)<c,$$

where c is an effectively computable constant depending only on $F(X)$ and a.

Our Corollary is a consequence of our Theorem and of the following powerful result from the theory of diophantine equations.

Lemma. *Let $t(X)\in\mathbb{Q}[X]$ and suppose that the polynomial $t(X)$ possesses at least three simple zeros. Then the equation*

$$t(x)=y^m \qquad \text{in integers } x,y,m \text{ with } |y|>1 \text{ and } m>1$$

implies that

$$\max(|x|,|y|,m)<c_1,$$

where c_1 is an effectively computable constant depending only on the polynomial $t(X)$.

PROOF. See Theorem 9.1 in [8].

Remark. Several special cases of the equation (1) have been considered and have been applied to certain combinatorial diophantine problems. The equations

$$\binom{x}{n} = \binom{y}{2} \quad \text{and} \quad \binom{x}{n} = \binom{y}{4} \tag{2}$$

lead to equations

$$8\binom{x}{n} + 1 = (2y-1)^2 \quad \text{and} \quad 24\binom{x}{n} + 1 = (y^2 - 3y - 1)^2, \text{ respectively.}$$

Applying the Corollary to the polynomials $F(X) = 8\binom{x}{n} + 1$ and $F(X) = 24\binom{x}{n} + 1$, respectively, we have that all the solutions of the equations (2) are bounded by an effectively computable constant depending only on n. By using another approach Kiss [6] (in the case n is a prime number) and Brindza [1] have proved that all the zeros of the polynomial $8\binom{x}{n} + 1$ are simple, thus the first equation of (2) has only finitely many solutions.

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ÁKOS PINTÉR
KOSSUTH LAJOS UNIVERSITY
MATHEMATICAL INSTITUTE
DEBRECEN, P.O.BOX 12.
4010 HUNGARY

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