

Metrizable linear connections in vector bundles

By MIHAI ANASTASIEI (Iași)

*Dedicated to Professor Dr. Lajos Tamássy
at his 80th anniversary*

Abstract. A linear connection ∇ in a vector bundle is said to be metrizable if the vector bundle admits a Riemannian metric h with the property $\nabla h = 0$. Sufficient conditions for the linear connection ∇ to be metrizable are provided.

Introduction

The problem of the metrizability of a linear connection was treated by many authors in various contexts (see the paper [7] by L. TAMASSY and the references therein). When a linear connection ∇ in a vector bundle $\xi = (E, p, M)$ is metrizable, its parallel translations are isometries with respect to any Riemannian metric h in ξ with $\nabla h = 0$. Using a local chart around a point x in M , the holonomy group $\phi(x)$ may be identified with a subgroup of $GL(m, \mathbb{R})$, where m is the dimension of fibre. With this identification, a necessary condition for ∇ to be metrizable is that the holonomy group be contained in the orthogonal group $O(m)$. We prove two versions of the converse of this fact (Theorems 3.1 and 3.2). Then we are dealing with the same problem when the vector bundle ξ is endowed with a Finsler function. The linear connection ∇ induces a

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nonlinear connection on E and a linear connection D in the vertical vector bundle over E . The Finsler function F defines a Riemannian metric g in the vertical vector bundle over E . We show that if g is covariant constant on horizontal directions, then ∇ is metrizable (Theorem 4.2). When the tangent bundle of a manifold M is endowed with a Finsler function F one says that (M, F) is a Finsler manifold. In this case our result is to be compared with the one due to Z. SZABÓ ([6]), regarding the metrizability of the Berwald connection.

If the cotangent bundle of a manifold M is endowed with a Finsler function K , then the pair (M, K) is called a *Cartan space*. This notion was introduced and studied by R. MIRON in [3]. In this case Theorem 4.1 is to be compared with our previous results on the metrizability of the Berwald–Cartan connection [1].

The first two sections of the paper are devoted to some preliminaries from the theory of vector bundles and linear connections in vector bundles.

1. Vector bundles

Let $\xi = (E, p, M)$ be a vector bundle of rank m . Here E and M are smooth i.e. C^∞ manifolds with $\dim M = n$, $\dim E = n+m$, and $p : E \rightarrow M$ is a smooth submersion. The fibres $E_x = p^{-1}(x)$, $x \in M$ are linear spaces of dimension m which are isomorphic with the type fibre $\mathbb{R}^m$.

Let $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ be an atlas on M . A vector bundle atlas is $\{(U_\alpha, \varphi_\alpha, \mathbb{R}^m)\}_{\alpha \in A}$ with the bijections $\varphi_\alpha : p^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^m$ in the form $\varphi_\alpha = (p(u), \varphi_{\alpha, p(u)}(u))$, where $\varphi_{\alpha, p(u)} : E_p(u) \rightarrow \mathbb{R}^m$ is a bijection. The given atlas on M and a vector bundle atlas provide an atlas $(p^{-1}(U_\alpha), \phi_\alpha)_{\alpha \in A}$ on E . Here $\phi_\alpha : p^{-1}(U_\alpha) \rightarrow \varphi_\alpha(U_\alpha) \times \mathbb{R}^m$ is the bijection given by $\phi_\alpha(u) = (\psi_\alpha(p(u)), \varphi_{\alpha, p(u)}(u))$. For $x \in M$, we put $\psi_\alpha(x) = (x^i) \in \mathbb{R}^m$ and we take (x^i, y^a) as local coordinates on E . If (U_β, ψ_β) is such that $x \in U_\alpha \cap U_\beta \neq \emptyset$ and $\psi_\beta(x) = (\tilde{x}^i)$, then $\psi_\beta \circ \psi_\alpha^{-1}$ has the form

$$\tilde{x}^i = \tilde{x}^i(x^1, \dots, x^n), \quad \text{rank} \left(\frac{\partial \tilde{x}^i}{\partial x^j} \right) = n. \quad (1.1)$$

Let (e_a) be the canonical basis of $\mathbb{R}^m$. Then $\varphi_{\alpha,x}^{-1}(e_a) = \varepsilon_a(x)$ is a basis of E_x and $u \in E_x$ takes the form $u = y^a \varepsilon_a(x)$. We put $\tilde{y}^a = M_b^a(x) y^b$ with $\text{rank}(M_b^a(x)) = m$. Then $\phi_\beta \circ \phi_\alpha^{-1}$ has the form

$$\begin{aligned} \tilde{x}^i &= \tilde{x}^i(x^1, \dots, x^n), & \text{rank} \left(\frac{\partial \tilde{x}^i}{\partial x^j} \right) &= n \\ \tilde{y}^a &= M_b^a(x) y^b, & \text{rank}(M_b^a(x)) &= m. \end{aligned} \quad (1.2)$$

The indices $i, j, k, \dots, a, b, c, \dots$ will take the values $1, 2, \dots, n$ and $1, 2, \dots, m$, respectively. The Einstein convention on summation will be used.

We denote by $\mathcal{F}(M), \mathcal{F}(E)$ the ring of real functions on M and E respectively, and by $\mathcal{X}(M)$, resp. $\Gamma(E)$, $\mathcal{X}(E)$ the module of sections of the tangent bundle of M , resp. of the bundle ξ and of the tangent bundle of E . On U_α , the vector fields $(\partial_k := \frac{\partial}{\partial x^k})$ provide a local basis for $\mathcal{X}(U_\alpha)$. The sections $\varepsilon_a : U_\alpha \rightarrow p^{-1}(U_\alpha)$ given by $\varepsilon_a(x) = \varphi_{\alpha,x}^{-1}(e_a)$ will be taken as canonical basis for $\Gamma(p^{-1}(U_\alpha))$ and a section $A : U_\alpha \rightarrow p^{-1}(U_\alpha)$ will take the form $A(x) = A^a(x) \varepsilon_a(x)$.

Let $\xi^* = (E^*, p^*, M)$ be the dual of the vector bundle ξ . We take as local basis of $\Gamma(E^*)$ on U_α the sections $\theta^a : U_\alpha \rightarrow p^{*-1}(U_\alpha)$, $x \rightarrow \theta^a(x) \in E_x^*$ such that $\theta^a(\varepsilon_b(x)) = \delta_b^a$.

Next, we may consider the tensor bundle of type (r, s) , $T_s^r(E) := E \otimes \dots \otimes E \otimes E^* \otimes \dots \otimes E^*$ over M and its sections. For $g \in \Gamma(E^* \otimes E^*)$

we have the local representation $g = g_{ab}(x) \theta^a \otimes \theta^b$. As $E^* \otimes E^* \cong L_2(E, \mathbb{R})$, we may regard g as a smooth mapping $x \rightarrow g(x) : E_x \times E_x \rightarrow \mathbb{R}$ with $g(x)$ a bilinear mapping given by $g(x)(s_a, s_b) = g_{ab}(x)$.

If the mapping $g(x)$ is symmetric i.e. $g_{ab} = g_{ba}$ and positive-definite i.e. $g_{ab}(x) \zeta^a \zeta^b > 0$ for every $0 \neq (\zeta^a) \in \mathbb{R}^m$, one says that g defines a Riemannian metric in the vector bundle ξ .

The sets of sections $\Gamma(T_s^r(E))$ are $\mathcal{F}(M)$ -modules for any natural numbers r, s . On the sum $\bigoplus_{r,s} \Gamma(T_s^r(E))$ a tensor product can be defined and one gets a tensor algebra $\mathcal{T}(E)$. For the vector bundle (TM, τ, M) this reduces to the tensor algebra of the manifold M .

2. Linear connections in a vector bundle

Definition 2.1. A linear connection in the vector bundle $\xi = (E, p, M)$ is a mapping $\nabla : \mathcal{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$, $(X, A) \rightarrow \nabla_X A$ which is $\mathcal{F}(M)$ -linear in the first argument, additive in the second and

$$\nabla_X(fA) = X(f)A + f\nabla_X A, \quad f \in \mathcal{F}(M). \quad (2.1)$$

For $X = X^k(x)\partial_k$ and $A = A^a(x)\varepsilon_a(x)$, we get

$$\nabla_X A = X^k(\partial_k A^a + \Gamma_{bk}^a(x)A^b)\varepsilon_a(x), \quad (2.2)$$

where the local coefficients $\Gamma_{bk}^a(x)$ are defined by

$$\nabla_{\partial_k}\varepsilon_b = \Gamma_{bk}^a\varepsilon_a. \quad (2.3)$$

If $\tilde{\Gamma}_{dj}^c$ are the local coefficients of ∇ on U_β such that $U_\alpha \cap U_\beta \neq \emptyset$, then we have

$$\tilde{\Gamma}_{dj}^c(\tilde{x}(x)) = M_a^c(x)(M^{-1})_d^b \frac{\partial x^k}{\partial \tilde{x}^j} \Gamma_{bk}^a(x) - \frac{\partial M_b^c}{\partial x^k} \frac{\partial x^k}{\partial \tilde{x}^j} (M^{-1})_d^b. \quad (2.4)$$

A section A of ξ is called *parallel* if $\nabla_X A = 0$ for every $X \in \mathcal{X}(M)$.

The linear connection ∇ induces operators of covariant derivative ∇_k in the tensor algebra $\mathcal{T}(E)$ taking $\nabla_k f = \partial_k f$, $\nabla_k \beta_a = \partial_k \beta_a - \Gamma_{ak}^c \beta_c$ and requiring that ∇_k to satisfy the Newton–Leibniz rule with respect to the tensor product and to commute with all contractions.

Let $c : [0, 1] \rightarrow M$ be a curve on M and $A : t \rightarrow A(t) := A(c(t))$ a section of ξ along the curve c . Then $\nabla_{\dot{c}(t)} A =: \frac{\nabla A}{dt}$ is called the covariant derivative of A along c .

On $U_\alpha \cap c[0, 1]$ if we put $c(t) = (x^i(t))$, we get

$$\frac{\nabla A}{dt} = \left(\frac{dA^a}{dt} + \Gamma_{bk}^a(x(t))A^b \frac{dx^k}{dt} \right) \varepsilon_a. \quad (2.5)$$

The section $t \rightarrow A(t)$ is said to be *parallel* on c if $\frac{\nabla A}{dt} = 0$. This means that the functions $(A^a(t))$ have to be solutions of the following system of ordinary linear differential equations:

$$\frac{dA^a}{dt} + \Gamma_{bk}^a(x)A^b \frac{dx^k}{dt} = 0. \quad (2.6)$$

For given initial conditions $A^a(0) = (u^a) \in E_{c(0)}$ the system (2.6) admits a unique solution that can be prolonged beyond U_α providing a parallel section A along c . If we associate to $(u^a) = A^a(0)$ the element $(v^a) = A^a(1) \in E_{c(1)}$ we get a linear isomorphism $P_c : E_{c(0)} \rightarrow E_{c(1)}$, called the *parallel translation* of $E_{c(0)}$ to $E_{c(1)}$ along c . The parallel translations can be defined along any curve or segment of curve providing linear isomorphisms between fibres in various points of curves on M . In particular, if one considers the loops with origin in $x \in M$, the corresponding parallel translations as linear isomorphisms $E_x \rightarrow E_x$ can be composed and a group $\phi(x)$ called the holonomy group in $x \in M$ is obtained.

When M is connected, the holonomy groups $\phi(x)$, $x \in M$ are isomorphic and one speaks about the holonomy group ϕ associated to or defined by ∇ .

The covariant derivative along c can be recovered from parallel translations according to the following known

Lemma 2.1. *Let A be a section of ξ along a curve on M , $c : t \rightarrow c(t)$, $t \in \mathbb{R}$, starting from $x = c(0)$. Then*

$$(\nabla_{\dot{c}(0)} A)(x) = \lim_{t \rightarrow 0} \frac{1}{t} (P_c(A(t)) - A(0)), \quad (2.7)$$

where $P_c : E_{c(t)} \rightarrow E_x$ is the parallel translation along c .

3. A sufficient condition for ∇ to be metrizable

Let ∇ be a linear connection in the vector bundle $\xi = (E, p, M)$. Assume that the manifold M is connected. One says that ∇ is *metrizable* if there exists a Riemannian metric g in ξ such that $\nabla g = 0$. When ∇ is metrizable, then all parallel translations $P_c : (E_x, g_x) \rightarrow (E_y, g_y)$ for any points x, y and for any curve c joining them in M are isometries. In particular, the holonomy group $\phi(x)$ is a subgroup of the orthogonal group of (E_x, g_x) . These facts follow from

Lemma 3.1. *Let g be any Riemannian metric in the vector bundle ξ and $c : t \rightarrow c(t)$, $t \in \mathbb{R}$, a curve in M with $c(0) = x$. Then*

$$(\nabla_{\dot{c}(0)}g)(A, B) = \lim_{t \rightarrow 0} \frac{1}{t} (g_{c(t)}(P_c A, P_c B) - g_x(A, B)), \quad (3.1)$$

where $A, B \in E_x$ and $P_c : E_x \rightarrow E_{c(t)}$ is the parallel translation along c .

PROOF. Let $\tilde{A}, \tilde{B}$ be sections of ξ which are parallel on c , such that $\tilde{A}(0) = A$, $\tilde{B}(0) = B$. Then $P_c A = \tilde{A}(t)$ and $P_c(B) = \tilde{B}(t)$. By the Taylor theorem and using the condition that $\tilde{A}$ and $\tilde{B}$ are parallel sections on c , in the natural basis (ε_a) we get $(P_c A)^a = \tilde{A}^a(t) = A^a + \frac{d\tilde{A}}{dt}(\tau)t = A^a - \Gamma_{ck}^a(x(\tau))\tilde{A}^k(\tau)\frac{dx^k}{dt}t$ and a similar formula for $(P_c B)^b$, $a, b = 1, 2, \dots, m$. Then, using again the Taylor theorem, omitting the terms which contain t^2 , we may write:

$$\begin{aligned} g_{ab}(t)(P_c A)^a(P_c B)^b - g_{ab}(x)A^a B^b &= \left(g_{ab}(x) + \frac{dg_{ab}}{dt}(\theta)t \right) (P_c A)^a(P_c B)^b \\ &- g_{ab}(x)A^a B^b = \left(\frac{dg_{ab}}{dt} - g_{ac}\Gamma_{bk}^c \frac{dx^k}{dt} - g_{cb}\Gamma_{ak}^c \frac{dx^k}{dt} \right) A^a B^b t, \end{aligned} \quad (3.2)$$

where the terms in the last paranthesis are computed for $\tau, \tau', \theta \in (0, t)$.

Dividing in (3.2) by t and taking $t \rightarrow 0$, one obtains (3.1).

By Lemma 3.1 we have also that if all parallel translations of ∇ are isometries with respect to g , then $\nabla g = 0$. Thus, in order to prove that ∇ is metrizable we need to find a Riemannian metric g such that all parallel translations of ∇ are isometries with respect to g . Taking an arbitrary bundle chart $(U_\alpha, \varphi_\alpha, \mathbb{R}^m)$, using the linear isomorphism $\varphi_{\alpha, x} : E_x \rightarrow \mathbb{R}^m$, we may identify $\phi(x)$, $x \in U_\alpha$, with a subgroup of $GL(\mathbb{R}^m)$. When ∇ is metrizable, by Lemma 3.1 it follows that this subgroup is contained in the orthogonal group $O(m)$. Therefore, a necessary condition for ∇ to be metrizable is that its holonomy group is contained in $O(m)$. We show two versions of the converse. $\square$

Theorem 3.1. *Let ∇ be a linear connection in the vector bundle $\xi = (E, p, M)$ with M connected. Assume that there exists a point $x_0 \in M$ such that the holonomy group $\phi(x_0)$ is contained in the orthogonal group of E_{x_0} when E_{x_0} is regarded as being isomorphic with the Euclidean space $(\mathbb{R}^m, \langle \cdot, \cdot \rangle)$ via a fixed bundle chart. Then ∇ is metrizable.*

PROOF. Let h_0 be the inner product on E_{x_0} induced by $\langle \cdot, \cdot \rangle$ via the bundle chart $(U_\alpha, \varphi_\alpha, \mathbb{R}^m)$, $x_0 \in U_\alpha$, that is,

$$h_0(u, v) = \langle \varphi_{\alpha, x_0} u, \varphi_{\alpha, x_0} v \rangle. \quad (*)$$

By hypothesis this inner product is invariant under the group $\phi(x_0)$. Let x be any point of M . We join x with x_0 using a curve $c : [0, 1] \rightarrow M$, $c(0) = x$, $c(1) = x_0$, consider the parallel translation $P_c : E_x \rightarrow E_{x_0}$ and define an inner product h_x in E_x by

$$h_x(A, B) = h_0(P_c A, P_c B), \quad A, B \in E_x. \quad (3.3)$$

Lemma 3.2. *The inner product h_x does not depend on the curve c .*

Indeed, if $\tilde{c}$ is another curve joining x with x_0 , then we consider the reverse c_- of c and the loop $\tilde{c} \circ c_-$ in x_0 . It follows that $h_0(P_{\tilde{c} \circ c_-} u, P_{\tilde{c} \circ c_-} v) = h_0(u, v)$, $u, v \in E_{x_0}$. Inserting here $u = P_c A$ and $v = P_c B$ and taking into account (3.3), the lemma follows.

The mapping $x \rightarrow h_x$ is smooth since P_c smoothly depends on x according to the general theory of differential equations. Thus we obtain a Riemannian metric h in ξ . The parallel translations of ∇ are isometries with respect to h . Indeed, for a point y of M different from x , any parallel translation from E_x to E_y has the form $P_{\sigma_- \circ c} = P_{\sigma_-} \circ P_c$, for σ_- the reverse of a curve σ joining y with x_0 . As a product of isometries this is an isometry. Therefore, using Lemma 3.1 we may conclude that $\nabla h = 0$. $\square$

The following version of Theorem 3.1 extends to the vector bundle setting a result of B. G. SCHMIDT [5].

Theorem 3.2. *Let ∇ be a linear connection in the vector bundle $\xi = (E, p, M)$ with M connected. Assume that for a fixed $x_0 \in M$, the holonomy group $\phi(x_0)$ leaves invariant a given positive-definite quadratic form h_0 on E_{x_0} . Then there exists a Riemannian metric h in ξ such that $\nabla h = 0$.*

PROOF. Let us denote by the same letter h_0 the inner product in E_{x_0} defined by the quadratic form h_0 . This inner product could be obtained by transferring one from $\mathbb{R}^m$ using a bundle chart. By hypothesis the inner

product h_0 is invariant under $\phi(x_0)$. From now on the reasoning proving Theorem 3.1 can be repeated in its entirety in order to find h such that $\nabla h = 0$. $\square$

Remark 3.1. The Riemannian metric h found in Theorem 3.1 is not unique and is not canonical in any way. The same applies for h found in Theorem 3.2.

4. Another condition for ∇ to be metrizable

We are to deal with the problem of the metrizability of a linear connection ∇ in a vector bundle endowed with a Finsler function.

Definition 4.1. Let $\xi = (E, p, M)$ be a vector bundle of rank m . A *Finsler function* on E is a nonnegative real function F on E with the properties

- 1) F is smooth on $E \setminus \{(x, 0), x \in M\}$,
- 2) $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$,
- 3) The matrix with the entries $g_{ab}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^a \partial y^b}$ is positive definite.

On the manifold E we have the vertical distribution $u \rightarrow V_u E = \ker p_{*,u}$ where p_* denotes the differential of p . This is spanned by $\dot{\partial}_a := \frac{\partial}{\partial y^a}$. A distribution $u \rightarrow H_u E$ which is supplementary to the vertical distribution is called a *horizontal* distribution or a *nonlinear connection* on E . This is usually taken as spanned by $\delta_i = \partial_i - N_i^a(x, y) \dot{\partial}_a$, where the functions $(N_i^a(x, y))$ are called the *coefficients* of the given nonlinear connection. Under a change of coordinates they behave as follows:

$$\tilde{N}_j^a \frac{\partial \tilde{x}^j}{\partial x^k} = M_b^a(x) N_k^b(x, y) - \frac{\partial M_b^a}{\partial x^k} y^b, \quad (4.1)$$

a fact which is equivalent to

$$\delta_i = \frac{\partial \tilde{x}^k}{\partial x^i} \tilde{\delta}_k. \quad (4.1')$$

Introducing the horizontal distribution we have

$$T_u E = H_u E \oplus V_u E, \quad u \in E. \quad (4.2)$$

It is convenient to decompose the geometrical objects on E according to (4.2) using the adapted basis $(\delta_i, \dot{\partial}_a)$ and its dual $(dx^i, \delta y^a = dy^a + N_i^a(x, y)dx^i)$.

The linear connection ∇ in ξ defines a nonlinear connection on E if we take $N_i^a(x, y) = \Gamma_{bi}^a(x)y^b$. Indeed, using (2.4) it is easy to check that these functions satisfy (4.1). From now on we shall use only the decomposition (4.2) provided by these functions. Furthermore, the linear connection ∇ induces a linear connection D in the vertical bundle over E as follows: $D : \mathcal{X}(E) \times \Gamma(VE) \rightarrow \Gamma(VE)$, $(X, Z) \rightarrow D_X Z$ is given for $Z = Z^a \dot{\partial}_a$ by

$$D_{\delta_k} \dot{\partial}_a = \Gamma_{bk}^a(x) \dot{\partial}_a, \quad D_{\dot{\partial}_b} \dot{\partial}_a = 0. \quad (4.3)$$

We call D the vertical lift of ∇ and we use D_{δ_k} for defining a *horizontal* covariant derivative operator in the tensor algebra of the vertical bundle, denoted by $|_k$, setting

$$\begin{aligned} f|_k &= \delta_k f \quad \text{for any function on } E, \\ X|_k &= \delta_k X^a + \Gamma_{bk}^a(x) X^b. \end{aligned} \quad (4.4)$$

For a fixed $x \in E$, the pair (E_x, F_x) is a Minkowski space. Here F_x denotes the restriction of F to E_x and it is obvious that this is a Minkowski norm on E_x .

Now we show that under certain conditions the parallel translations of ∇ are isometries of Minkowski spaces.

Theorem 4.1. *Let $\xi = (E, p, M)$ be a vector bundle of rank m with M connected, endowed with a Finsler function F and with a linear connection ∇ as well. Let $|_k$ be the horizontal covariant derivative operator defined by the vertical lift D of ∇ . If $F|_k = 0$, then the parallel translation defined by ∇ , $P_c : (E_x, F_x) \rightarrow (E_y, F_y)$ is an isometry of Minkowski spaces for any points $x, y \in M$ and any curve $c : [0, 1] \rightarrow M$ joining them.*

PROOF. Let be $u \in E_x$, and $t \rightarrow A(t)$, $t \in [0, 1]$ a section of ξ which is parallel along c , and $A(0) = u$. Its local components A^a are solutions of the system of differential equations (2.6), and $P_c(u) = A(1) := v$.

We know already that P_c is a linear isomorphism. Let us write out the condition $F|_k = 0$ for the points $(x(t), A(t))$ of E where $t \rightarrow x(t)$ is the

local representation of the curve c . We obtain:

$$0 = \left(\frac{\partial F}{\partial x^k} - A^b \Gamma_{bk}^a \frac{\partial F}{\partial y^a} \right) \frac{dx^k}{dt} \stackrel{(2.6)}{=} \frac{\partial F}{\partial x^k} \frac{dx^k}{dt} + \frac{\partial F}{\partial y^a} \frac{dA^a}{dt} = \frac{dF(x(t), A(t))}{dt}.$$

Thus the function $F(x(t), A(t))$ is constant. It follows $F(x, u) = F(y, P_c u)$, that is, $F_x(u) = F_y(P_c u)$. In other words, P_c is an isometry of Minkowski spaces (E_x, F_x) and (E_y, F_y) . $\square$

Corollary 4.1. *Under the hypothesis of Theorem 4.1, the holonomy group $\phi(x)$ consists of isometries of the Minkowski space (E_x, F_x) .*

The functions $g_{ab}(x, y)$ define a Riemannian metric in the vertical bundle over E by $g = g_{ab}(x, y) \delta y^a \otimes \delta y^b$. We call $(g_{ab}(x, y))$ the Finsler metric associated with F .

The condition $F|_k = 0$ from the hypothesis of Theorem 4.1 can be replaced by $g_{ab}|_k = 0$, because of

Lemma 4.1. *$F|_k = 0$ is equivalent to $g_{ab}|_k = 0$.*

PROOF. The homogeneity of F implies $F^2(x, y) = g_{ab}(x, y) y^a y^b$. Then $F^2|_k = 2FF|_k = g_{ab}|_k y^a y^b + 2g_{ab} y|_k^a y^b = g_{ab}|_k y^a y^b$ since $y|_k^a = 0$. Thus if $g_{ab}|_k = 0$, then $F|_k = 0$. In order to prove the converse, we notice that $\dot{\partial}_a(H|_k) = (\dot{\partial}_a H)|_k$ for any function H on E . This follows by a direct calculation taking into account that $\dot{\partial}_a H$ is a vertical 1-form. Using this “commutation” formula we get $g_{ab}|_k = \frac{1}{2} \dot{\partial}_a \dot{\partial}_b (F^2|_k) = \dot{\partial}_a \dot{\partial}_b (FF|_k) = 0$. $\square$

Now we are ready to prove the main result of this section.

Theorem 4.2. *Let ∇ be a linear connection in the vector bundle $\xi = (E, p, M)$ with M connected. Suppose that E is endowed with a Finsler function F having the associated Finsler metric $g_{ab}(x, y)$. Let $|_k$ be the h -covariant derivative operator induced by ∇ . If $g_{ab}|_k = 0$, then ∇ is metrizable.*

PROOF. For a fixed $x_0 \in M$ we have the Minkowski space (E_{x_0}, F_{x_0}) . Let G be the group of all linear isomorphisms of E_{x_0} which preserve the set $S_{x_0} = \{u \in E_{x_0}, F_{x_0}(u) = 1\}$. This G is a compact Lie group since S_{x_0} is compact. In our hypothesis, according to Lemma 4.1 and Corollary 4.1,

the holonomy group $\phi(x_0)$ is a Lie subgroup of G . Let $\langle \cdot, \cdot \rangle$ be any inner product on E_{x_0} . Define a new inner product on E_{x_0} by

$$h_{x_0}(u, v) = \frac{1}{\text{vol}(G)} \int_G \langle gu, gv \rangle \mu_G, \quad (4.5)$$

for $u, v \in E_{x_0}$, $g \in G$ and μ_G the bi-invariant Haar measure on G .

It follows that for every $a \in G$ we have

$$h_{x_0}(au, av) = h_{x_0}(u, v), \quad u, v \in E_{x_0}. \quad (4.6)$$

In particular, (4.6) holds for any element of $\phi(x_0) \subset G$. Thus $\phi(x_0)$ leaves invariant the inner product h_{x_0} in E_{x_0} . The inner product h_{x_0} is extended by parallel translations to a Riemannian metric h in ξ . Furthermore, this metric verifies $\nabla h = 0$ since all parallel translations of ∇ become isometries with respect to h . Thus ∇ is metrizable. $\square$

Remark 4.1. The Riemannian metric h is not unique and it is not canonical in any way.

References

- [1] M. ANASTASIEI, Geometry of Berwald–Cartan spaces (*to appear*).
- [2] R. MIRON, Techniques of Finsler geometry in the theory of vector bundles, *Acta Sci. Math.* **49** (1985), 119–129.
- [3] R. MIRON, Hamilton geometry, *Univ. Timișoara (Romania), Sem. Mecanica* **3** (1987), 54.
- [4] R. MIRON and M. ANASTASIEI, The Geometry of Lagrange Spaces: Theory and Applications, *Kluwer Academic Publishers*, 1994, FTPH 59.
- [5] B. G. SCHMIDT, Conditions on a Connection to be a Metric Connection, *Commun. Math. Phys.* **29** (1973), 55–59.
- [6] Z. I. SZABÓ, Positive Definite Berwald Spaces, *Tensor N.S.* **35** (1981), 25–39.
- [7] L. TAMASSY, Metrizable of Affine Connections, *Balkan J. of Geometry and Its Applications* **1**, no. 1 (1996), 83–90.

MIHAI ANASTASIEI
 FACULTY OF MATHEMATICS
 UNIVERSITY “AL.I.CUZA” IAȘI
 6600, IAȘI
 ROMANIA
 E-mail: anastas@uaic.ro

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