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Computer algebra and two and three dimensional Finsler geometry

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To Professor L. Tamássy on the occasion of his 80th anniversary

Abstract. After a review of 2- and 3-dimensional Finsler spaces from the Berwald and Cartan connections points-of-view, several tensors are explicitly worked out by means of Computer Algebra, and Moór frames are also used to obtain information about the *almost flat* metric

$$ds^2 = dr^2 + r^2 d\Omega^2 - dt^2 + \varepsilon d\Omega dt,$$

derived from the Rutz's 2 parameter family of metrics, an non-Riemannian first-order perturbation of the Schwarzschild solution (to Eintein's field equations), which solves, up to order ε, a generalized field equation consisting of a trace-free deviation tensor, $B^i_i = 0$. It is determined that such spaces are *not Landsberg*, are of *zero curvature* and yet are *not projectively flat*, all results being up to order ε. Also, only one of the three Cartan's curvature tensor turns out to be null, namely $S_{ijkl} = 0(\varepsilon^2)$, and the link to Brickell's theorem is mentioned. All calculations are performed using the FINSLER package, based on MAPLE.

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1. Preliminaries
(Finsler spaces and related geometric objects)

In this section we present the most important objects induced by a Finsler space of dimension n. We shall use these in the last two sections for the particular cases of dimension two and three, respectively.

We start with a real, smooth, n-dimensional manifold M, we denote by (TM, π, M) its tangent bundle, and by $\widetilde{TM} = TM\backslash\{0\}$ the tangent bundle with zero section removed. Every local chart $\big(U, \varphi = (x^i)\big)$ of the C^∞-structure on M, induces a local chart $\big(\pi^{-1}(U), \phi = (x^i, y^i)\big)$ on TM. As $\pi : TM \mapsto M$ is a submersion, the kernel of the linear map induced by it determines a regular n-dimensional distribution $V : u \in TM \mapsto V_u = \mathrm{Ker}(\pi_{*,u}) \subset T_uTM$. We call it the *vertical distribution*. If $\big\{\frac{\partial}{\partial x^i}\big|_u, \frac{\partial}{\partial y^i}\big|_u\big\}$ is the natural basis of T_uTM, then $\big\{\frac{\partial}{\partial y^i}\big|_u\big\}$ is a basis for $V_u, \forall u \in TM$. Consider $J = \frac{\partial}{\partial y^i} \otimes dx^i$, the *almost tangent structure* (J is also called the vertical endomorphism of TM), and $\Gamma = y^i \frac{\partial}{\partial y^i}$ the Liouville vector field. A vector field S on TM is called a *semispray* (or a second order vector field) if $JS = \Gamma$. The local expression of a semispray is $S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$. The functions $G^i(x, y)$ are called the *local coefficients* of the semispray and these are defined on a domain of an induced local chart. If G^i are homogeneous of degree two with respect to y then S is called a *spray*. This is equivalent with $[\Gamma, S] = S$.

An n-dimensional distribution $N : u \in TM \mapsto N_u \subset T_uTM$ that is supplementary to the vertical distribution V is called a *nonlinear connection* (or horizontal distribution). For every $u \in TM$ a nonlinear connection N induces the direct sum

$$T_uTM = N_u \oplus V_u. \tag{1.1}$$

An adapted basis to the previous sum is $\big\{\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}\big\}$. The functions $N_j^i(x, y)$ are defined on a domain of an induced local chart, and they are called the *local coefficients* of the nonlinear connection N. It is well known that every semispray S with local coefficients G^i, induces a nonlinear connection N with local coefficients $N_j^i = \frac{\partial G^i}{\partial y^j}$, [7].

Definition 1.1. A Finsler space is a pair (M, F), where F is a positive-valued function defined on TM, such that:

1° F is of C^∞-class on $\widetilde{TM}$, and continuous on the null section;

2° F is positively homogeneous of degree one with respect to y, that is $F(x, \lambda y) = |\lambda| F(x, y)$, $\forall \lambda \in \mathbb{R}$;

3° The matrix with the entries $g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is non–degenerate on $\widetilde{TM}$.

We call (g_{ij}) the metric tensor of the Finsler space, it is zero homogeneous with respect to y and we have that $F^2(x, y) = g_{ij}(xy) y^i y^j$. A very important tensor on Finsler geometry is

$$C_{ijk} = \frac{1}{4} \frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}. \tag{1.2}$$

It is called the *Cartan tensor*, it is totally symmetric and it vanishes if and only if the metric tensor (g_{ij}) depends only on x, that is the Finsler space reduces to a Riemannian space. Denote by $C^i_{jk} = g^{i\ell} C_{\ell jk}$. Throughout this paper, we shall rise and down indices using the Finsler metric g_{ij}, and we shall use the summation convention on upper and lower repeated indices. Consider $C^i = C^i_{jk} g^{jk}$, the *Cartan vector*. As $C_{ijk} y^i = 0$ we have that $C^i y_i = C^i y^j g_{ij} = 0$, so the Cartan vector $C^i \frac{\partial}{\partial y^i}$ and the Liouville vector $y^i \frac{\partial}{\partial y^i}$ are g-orthogonal.

For a Finsler space, there is a canonical spray S, with local coefficients (G^i) given by [11]

$$2G^i = \frac{1}{2} g^{ij} \left(\frac{\partial^2 F^2}{\partial y^j \partial x^m} y^m - \frac{\partial F^2}{\partial x^j} \right). \tag{1.3}$$

Next we shall use this canonical spray and the induced nonlinear connection. With respect to this, the spray can be expressed as $S = y^i \frac{\delta}{\delta x^i}$, that is S is a horizontal vector field.

A tensor field of (r, s)-type $\left(T^{i_1, \ldots, i_r}_{j_1, \ldots, j_s}(x, y)\right)$ is called a *Finsler tensor* field [3] (or a d-tensor field) if under a change of induced local coordinates on TM, it transforms as a (r, s)-type tensor field on the base manifold M. The tensor fields we have met so far are Finsler tensor fields: (g_{ij}) is a Finsler tensor field of $(0, 2)$-type, C_{ijk} of $(0, 3)$-type and C^i of $(1, 0)$-type.

Definition 1.2. A linear connection $\mathcal{D} : (X, Y) \in \mathcal{X}(TM) \times \mathcal{X}(TM) \mapsto \mathcal{D}_X Y \in \mathcal{X}(TM)$ on TM is called a *Finsler connection* (or a d-connection) if it preserves by parallelism the horizontal distribution and the almost tangent structure is absolutely parallel with respect to it.

Denote by h and v, the horizontal and vertical projectors that correspond to the direct sum (1.1) and the canonical nonlinear connection. So, a linear connection $\mathcal{D}$ on TM is a Finsler connection if and only if $\mathcal{D}h = 0$ and $DJ = 0$. It can be proved very easily that a Finsler connection $\mathcal{D}$ preserves also by parallelism the vertical distribution, that is $\mathcal{D}v = 0$. With respect to the basis $\left\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right\}$, adapted to the decomposition (1.1), a Finsler connection can be expressed as follows:

$$\begin{cases} \mathcal{D}_{\frac{\delta}{\delta x^i}} \dfrac{\delta}{\delta x^j} = F^k_{ji} \dfrac{\delta}{\delta x^k}, \quad \mathcal{D}_{\frac{\delta}{\delta x^i}} \dfrac{\partial}{\partial y^j} = F^k_{ji} \dfrac{\partial}{\partial y^k}, \\ \mathcal{D}_{\frac{\partial}{\partial y^i}} \dfrac{\delta}{\delta x^j} = V^k_{ji} \dfrac{\delta}{\delta x^k}, \quad \mathcal{D}_{\frac{\partial}{\partial y^i}} \dfrac{\partial}{\partial y^j} = V^k_{ji} \dfrac{\partial}{\partial y^k}. \end{cases} \tag{1.4}$$

We may remark here that the horizontal coefficients $\left(F^i_{jk}(x,y)\right)$ of a Finsler connection transform like the coefficients of a linear connection on the base manifold M. The vertical coefficients $\left(V^i_{jk}(x,y)\right)$ are the components of a (1,2)-type Finsler tensor field.

If $\left(T^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}(x,y)\right)$ are the local components of a (r,s)-type Finsler tensor field, then we define the h-covariant derivative as [8]

$$\begin{aligned} T^{i_1,\ldots,i_r}_{j_1,\ldots,j_s|k} = {} & \frac{\delta T^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}}{\delta x^k} + T^{pi_2\ldots i_r}_{j_1,\ldots,j_s} F^{i_1}_{pk} + \cdots + T^{i_1,\ldots,i_{r-1}p}_{j_1,\ldots,j_s} T^{i_r}_{pk} \\ & - T^{i_1,\ldots,i_r}_{pj_2,\ldots,j_s} F^p_{j_1,k} - \cdots - T^{i_1,\ldots,i_r}_{j_1,\ldots,j_{s-1}p} F^p_{j_s k}, \end{aligned} \tag{1.5}$$

and the v covariant derivative as

$$\begin{aligned} T^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}\Big|_k = {} & \frac{\partial T^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}}{\partial y^k} + T^{pi_2\ldots i_r}_{j_1,\ldots,j_s} V^{i_1}_{pk} + \cdots + T^{i_1,\ldots,i_{r-1}p}_{j_1,\ldots,j_s} V^{i_r}_{pk} \\ & - T^{i_1,\ldots,i_r}_{pj_2,\ldots,j_s} V^p_{j_1 k} - \cdots - T^{i_1,\ldots,i_r}_{j_1,\ldots,j_{s-1}p} V^p_{j_s k}. \end{aligned} \tag{1.5'}$$

We may remark here that the h- and v-covariant derivative of a (r,s)-type Finsler tensor field is a $(r,s+1)$-type Finsler tensor field. The h-covariant derivative of the Liouville vector (y^i) is denoted by $d^i_j := y^i_{|j} = F^i_{kj} y^k - N^i_j$ and is called the *deflection tensor*.

The dynamical covariant derivative of a Finsler vector field $\left(X^i(x,y)\right)$ is defined by [11]

$$\nabla X^i = S(X^i) + N^i_j X^j = \frac{\partial X^i}{\partial x^j} y^j - 2\frac{\partial X^i}{\partial y^j} G^j + \frac{\partial G^i}{\partial y^i} X^j. \tag{1.6}$$

If the deflection tensor of a Finsler connection is zero, then the dynamical covariant derivative can be expressed as follows:

$$\nabla X^i = S(X^i) + F^i_{kj} y^k X^j = X^i_{|k} y^k. \tag{1.6}'$$

With respect to the basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ a Finsler connection has five components of torsion ([3], [11]), $T^k_{ij} = F^k_{ji} - F^k_{ij}$, $R^k_{ij} = \frac{\delta N^i_i}{\delta x^j} - \frac{\delta N^k_j}{\delta x^i}$, C^k_{ji}, $P^k_{ji} = \frac{\partial N^i_j}{\partial y^i} - F^k_{ji}$, and $S^k_{ij} = V^k_{ji} - V^k_{ij}$. It should be noted that R^k_{ij} is called also the curvature of the nonlinear connection because it is the obstruction from being integrable for the nonlinear connection $\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = -R^k_{ij}\frac{\partial}{\partial y^k}$.

The only three components of the curvature of a Finsler connection are given by [3], [10]

$$\begin{cases} R^i_{jk\ell} = \dfrac{\delta F^i_{jk}}{\delta x^\ell} - \dfrac{\delta F^i_{j\ell}}{\delta x^k} + F^m_{jk} F^i_{m\ell} - F^m_{j\ell} F^i_{mk} + V^i_{jm} R^m_{k\ell}; \\ P^i_{jk\ell} = \dfrac{\partial F^i_{jk}}{\partial y^\ell} - V^i_{jk|\ell} + V^i_{jm} P^m_{k\ell}; \\ S^i_{jk\ell} = \dfrac{\partial V^i_{jk}}{\partial y^\ell} - \dfrac{\partial V^i_{j\ell}}{\partial y^k} + V^m_{jk} V^i_{m\ell} - V^m_{j\ell} V^i_{mk}. \end{cases} \tag{1.7}$$

The most important Finsler connections in Finsler geometry are ([3], [9], [11]) the Cartan connection, the Berwald connection, the Chern–Rund connection, and the Hashiguchi connection. In this paper we shall deal only with Cartan and Berwald connections.

The Cartan connection is perfectly determined by the axioms [9]:

1° It is metric: $g_{ij|k} = 0$ and $g_{ij}|_k = 0$,

2° It is symmetric: $T^i_{jk} = 0$ and $S^i_{jk} = 0$;

3° The deflection tensor d^i_j vanishes.

The horizontal coefficients of the Cartan connection are given by [11]

$$F^i_{jk} = \frac{1}{2} g^{i\ell} \left(\frac{\delta g_{\ell j}}{\delta x^i} + \frac{\delta g_{\ell k}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^j} \right). \tag{1.8}$$

The vertical coefficients of the Cartan connection are $V^i_{jk} = C^i_{jk}$, where C^i_{jk} is the Cartan tensor.

The local coefficients of the Berwald connection are $F^i_{jk} = \frac{\partial N^i_j}{\partial y^k} = \frac{\partial^2 G^i}{\partial y^j \partial y^k}$ and $V^i_{jk} = 0$. For the Berwald connection there are two important properties: $F_{|i} = 0$ and the horizontal tensor of deflection is zero.

If $(R^i_{jk\ell})$ is the h-curvature component of either Cartan or Berwald connection, defined by (1.7), then $R^i_{jk\ell} y^j = R^i_{k\ell}$.

2. Geodesic deviation

The general KCC-theory [1] of a system of SODE has been studied by many authors [1], [21]. In this section we present this theory in the particular case when the system of SODE is induced by a Finsler space.

We consider (M, F) a Finsler space and its canonical spray S. The local coefficients of S are given by (1.3). A smooth curve $c : t \in I \mapsto c(t) = \big(x^i(t)\big) \in M$ is called a *path* of the spray S if its lift to TM, $\widetilde{c} : t \in I \mapsto \widetilde{c}(t) = \big(x^i(t), \frac{dx^i}{dt}(t)\big) \in TM$ is an integral curve of S. In local coordinates, a curve $c(t) = \big(x^i(t)\big)$ is a path of S if and only if

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0. \tag{2.1}$$

As the local coefficients of the spray are homogeneous of degree two with respect to y, then $\frac{\partial G^i}{\partial y^j} y^j = 2G^i$, that is equivalent to $2G^i = N^i_j y^j$. So, the equation (2.1) is equivalent to

$$\frac{d^2x^i}{dt^2} + N^i_j\left(x, \frac{dx}{dt}\right)\frac{dx^j}{dt} = 0. \tag{2.1'}$$

The equivalence of the systems (2.1) and (2.1)' says that a curve c is a path of the spray S if and only if its lift $\widetilde{c}$ is a horizontal curve. Also, as the h-deflection tensor of the Cartan and Berwald connection is zero, then $N^i_j = F^i_{jk} y^k$. Consequently the system (2.1)' is equivalent to

$$\frac{d^2x^i}{dt^2} + F^i_{jk}\left(x, \frac{dx}{dt}\right)\frac{dx^j}{dt}\,\frac{dx^k}{dt} = 0. \tag{2.1''}$$

Moreover the system (2.1)" is equivalent with $\mathcal{D}_{\dot{\widetilde{c}}}\dot{\widetilde{c}} = 0$, where $\dot{\widetilde{c}}$ is the tangent vector to the lift $\widetilde{c}$ of the curve c. This means that a curve c on M

is a path of the spray S if and only if $\widetilde{c}$ is a geodesic of the Cartan or Berwald connection.

An equivalent invariant form of the system (2.1) is

$$\nabla\left(\frac{dx^i}{dt}\right) = 0. \tag{2.2}$$

Now let $c(t) = \big(x^i(t)\big)$ be a path of the spray S and a variation of it into nearby ones according to

$$\widetilde{x}^i(t) = x^i(t) + \varepsilon\xi^i(t). \tag{2.3}$$

Here ε denotes a scalar parameter with small value $|\varepsilon|$, and $\xi^i(t)$ are components of a contravariant vector field along $c(t)$. If we substitute (2.3) into (2.1), ask for $\widetilde{c}(t) = \big(\widetilde{x}^i(t)\big)$ to be also a path of the spray and let $\varepsilon \to 0$, we get the so-called *variational equations*

$$\frac{d^2\xi^i}{dt^2} + 2\frac{\partial G^i}{\partial x^j}\xi^j + 2\frac{\partial G^i}{\partial y^j}\,\frac{d\xi^j}{dt} = 0. \tag{2.4}$$

For the variational equation (2.4) we have the equivalent invariant form (Jacobi equation)

$$\nabla^2\xi^i + B^i_j\xi^j = 0, \quad \text{where} \tag{2.5}$$

$$B^i_j := 2\frac{\partial G^i}{\partial x^j} - S\left(\frac{\partial G^i}{\partial y^j}\right) - \frac{\partial G^i}{\partial y^k}\,\frac{\partial G^k}{\partial y^j} = R^i_{jk}y^k. \tag{2.6}$$

Here (R^i_{jk}) is the curvature of the nonlinear connection induced by canonical spray S. As $R^i_{jk} = R^i_{\ell jk}y^\ell$, then $B^i_j = R^i_{\ell jk}y^\ell y^k$. B^i_j is called the second invariant of the given SODE, [1] or the Jacobi endomorphism.

Definition 2.1. A Lie symmetry of the spray S is a vector field X on the base manifold M such that $[S, X^c] = 0$, where X^c is the complete lift of X and is defined by

$$\text{if} \quad X = X^i(x)\,\frac{\partial}{\partial x^i}, \quad \text{then} \quad X^c = X^i(x)\,\frac{\partial}{\partial x^i} + \frac{\partial X^i}{\partial x^j}y^i\,\frac{\partial}{\partial y^j}. \tag{2.7}$$

Theorem 2.1 ([1])**.** *1. A vector field $X = X^i(x)\frac{\partial}{\partial x^i}$ is a Lie symmetry of S if and only if*

$$\nabla^2 X^i + B^i_j X^j = 0. \tag{2.8}$$

2. If $X = X^i(x)\,\frac{\partial}{\partial x^i}$ is a Lie symmetry of S and $c(t) = \big(x^i(t)\big)$ is a path of S, then the restriction of X along c is a Jacobi vector field.

From (1.6)' we have that $\nabla X^i = X^i_{|j} y^j$ and because $y^i_{|j} = 0$, then the equation (2.8) is equivalent to

$$(X^i_{|j|k} + R^i_{jmk} X^m) y^j y^k = 0. \tag{2.8}'$$

If we denote by $B_{ij} = g_{ik} B^k_j$, then $B_{ij} = R_{\ell ijk} y^\ell y^k$, and consequently B_{ij} is symmetric with respect to i and j.

Let $R_{\ell ijk}$ the h-curvature component of the Berwald or the Cartan connection. We define, for a Finsler tensor field $X^i(x, y)$, the flag curvature [5]

$$R(x, y, X) = \frac{R_{\ell ijk} y^\ell y^k X^i X^j}{(g_{\ell k} g_{ij} - g_{\ell i} g_{jk}) y^\ell y^k X^i X^j}. \tag{2.9}$$

Denote by $\ell_i = \frac{1}{F} y_i$ the normalized supporting element and $h_{ij} = F \frac{\partial^2 F}{\partial y^i \partial y^j}$ the angular metric. It is very easy to prove that the metric tensor and the angular metric are related by $g_{ij} = h_{ij} + \ell_i \ell_j$. Then we can rewrite (2.9) as

$$B_{ij}(x, y) X^i X^j = R(x, y, X) F^2(x, y) h_{ij}(x, y) X^i(x, y) X^j(x, y). \tag{2.10}$$

Definition 2.2. A Finsler space is said to have scalar curvature $R(x, y)$ if the flag curvature (2.9) does not depend on (X^i) at every point (x, y). From (2.10) we have that a Finsler space has scalar curvature if and only if the Jacobi endomorphism can be written as

$$B_{ij} = R F^2 h_{ij}. \tag{2.11}$$

If we denote by $h^i_j = g^{ik} h_{kj}$, then the Jacobi endomorphism of a Finsler space with scalar curvature can be written as

$$B^i_j = R F^2 h^i_j. \tag{2.11}'$$

Consequently the Jacobi equation of a Finsler space with scalar curvature, has the form

$$\nabla^2 \xi^i + R F^2 h^i_j \xi^j = 0. \tag{2.12}$$

3. Two dimensional Finsler space

It is well known that every two-dimensional Finsler space has scalar curvature [9]. Also there is a canonical frame, called the Berwald frame. We shall use this to express the KCC-invariants and study the geodesic deviation.

Consider (M, F^2) be a two-dimensional Finsler space. Denote by C^i the length of the Cartan vector C^i and $m^i = \frac{1}{C}C^i$. Then (ℓ^i, m^i) are unitary vector fields and because $\ell_i m^i = g_{ij}\ell^i m^j = 0$, then (ℓ^i, m^i) is an orthonormal frame, [6]. We call it the Berwald frame of the two-dimensional Finsler space (M, F^2).

With respect to the Berwald frame, the metric tensor g_{ij} and the angular tensor h_{ij} are given by

$$g_{ij} = \ell_i\ell_j + m_i m_j, \quad h_{ij} = m_i m_j. \tag{3.1}$$

Proposition 3.1. *The local coefficients of the Cartan connection are given by*

$$\begin{cases} F^i_{jk} = -\ell_j \dfrac{\delta \ell^i}{\delta x^k} - m_j \dfrac{\delta m^i}{\delta x^k}, \\ C^i_{jk} = -\ell_j \dfrac{\partial \ell^i}{y^k} - m_j \dfrac{\partial m^i}{\partial y^k} + \dfrac{1}{F}\ell_j m^i m_k + \dfrac{1}{F} m_j \ell^i m_k. \end{cases} \tag{3.2}$$

Proof. It is known [9] that the h- and v-covariant derivatives of the Berwald frame with respect to the Cartan connection are given by

$$\begin{cases} \ell^i_{|j} = 0; & m^i_{|j} = 0; \\ \ell^i|_j = \dfrac{1}{F} m^i m_j; & m^i|_j = -\dfrac{1}{F}\ell^i m_j. \end{cases} \tag{3.3}$$

□

If we solve these for F^i_{jk} and C^i_{jk}, then we get (3.2).

Proposition 3.2. *The dynamical covariant derivative of the Berwald frame vanishes identically, that is*

$$\nabla m_i = 0, \quad \nabla \ell_i = 0. \tag{3.4}$$

PROOF. According to (1.6)' the dynamical covariant derivative of ℓ_i is given by $\nabla\ell_i = \ell_{i|j}y^j$, and if we use (3.3) we have (3.4). □

Proposition 3.2 says that the Berwald frame is parallel along any path of the canonical spray.

Let (X^i) be a Finsler vector field, we denote by $(X^{(i)})$ its scalar component with respect to the Berwald frame, that is, $X^i = X^{(1)}\ell^i + X^{(2)}m^i$. Then the dynamical covariant derivative of X^i is given by

$$\begin{gathered}\nabla X^i = (X^{(1)})'\ell^i + (X^{(2)})'m^i, \quad \text{where } (X^{(1)})' = S(X^{(1)}) \\ \text{and} \quad (X^{(2)})' = S(X^{(2)}).\end{gathered} \tag{3.5}$$

As every two dimensional Finsler space is of scalar curvature we have that the Jacobi endomorphism has the form (2.11)', where $R = R_{1212}$.

Theorem 3.1. *The scalar form of the Jacobi equation* (2.5)*, in a two dimensional Finsler space with respect to the Berwald frame is given by*

$$\begin{cases}(X^{(1)})'' = 0, \\ (X^{(2)})'' + RF^2X^{(2)} = 0.\end{cases} \tag{3.6}$$

PROOF. Let X^i be a Jacobi vector field and denote by $(X^{(i)})$ its scalar components with respect to the Berwald frame. We have then $\nabla^2X^i = (X^{(1)})''\ell^i + (X^{(2)})''m^i$. Also, as $B^i_j = RF^2h^i_j$, and $h^i_j = m^im_j$, then the Jacobi equation (2.6) is equivalent to

$$(X^{(1)})''\ell^i + (X^{(2)})''m^i + RF^2m^im_j(X^{(1)}\ell^j + X^{(2)}m^j) = 0 \quad \Longleftrightarrow$$

$(X^{(1)})''\ell^i + (X^{(2)})''m^i + RF^2m^iX^{(2)} = 0$ and this is equivalent to (3.6). The equation (3.6) appears also in RUND's book [13, p. 114], but the way he gets it is very complicated.

From (3.6) we can see that the tangential component of every Jacobi vector field is of the form $(at + b)\ell^i$, while the orthogonal component is $X^{(2)}m^i$, where $X^{(2)}$ is the solution of $(3.6)_2$. So if X is a Jacobi vector field, then the length squared $\|X(t)\|^2 = (at+b)^2 + \left(X^{(2)}(t)\right)^2$, where $X^{(2)}$ is the solution of $(3.6)_2$. We can deduce from this that if $R > 0$ then the geodesics are stable (in other words the geodesic rays are bunching together) and if $R \leq 0$ then the geodesics are unstable (in other words the

geodesic rays are dispersing). See [4] for more discussion about geodesic stability.

We have to remark here that we can use also (3.6) to find out the Lie symmetries of a two dimensional Finsler space. So a vector field (X^i) is a Lie symmetry if and only if its scalar components $(X^{(i)})$ satisfy (3.6). □

4. Three dimensional Finsler space

Consider (M, F^2) a three dimensional Finsler space. As for a two-dimensional Finsler space we consider $\ell^i = \frac{1}{F}y^i$ and $m^i = \frac{1}{C}C^i$. Consider then (n^i) the unitary vector field orthogonal to (ℓ^i) and (m^i). Then (ℓ^i, m^i, n^i) is an orthogonal frame, [8]. We call it the Moór frame of the Finsler space. With respect to it, the tensor metric and the angular metric are expressed as follows:

$$g_{ij} = \ell_i\ell_j + m_im_j + n_in_j, \quad \text{and} \quad h_{ij} = m_im_j + n_in_j. \tag{4.1}$$

Denote by $(\ell_i^\alpha) = (\ell_i, m_i, n_i)$ and $(\ell_\alpha^i) = (\ell^i, m^i, n^i)$. Then we have that

$$\ell_i^\alpha \ell_\alpha^j = \delta_i^j, \quad \text{and} \quad \ell_\alpha^i \ell_i^\beta = \delta_\alpha^\beta. \tag{4.2}$$

Consider the vector fields $H_\alpha = \ell_\alpha^i \frac{\delta}{\delta x^i}$ and $V_\alpha = \ell_\alpha^i \frac{\partial}{\partial y^i}$. Denote by $F_{\beta\gamma}^\alpha$ and $V_{\beta\gamma}^\alpha$ the coefficients of a Finsler connection with respect to (H_α, V_α). We call these the scalar coefficients of the Finsler connection.

Proposition 4.1 ([2]). *Let $\mathcal{D}$ be a Finsler connection with local coefficients (F_{jk}^i, V_{jk}^i). Then, the scalar coefficients of the Finsler connection $\mathcal{D}$ are given by*

$$\begin{cases} F_{\alpha\beta}^\gamma = \ell_k^\gamma \ell_{\alpha|i}^k \ell_\beta^i = -\ell_{j|i}^\gamma \ell_\alpha^j \ell_\beta^i, \\ V_{\alpha\beta}^\gamma = \ell_k^\gamma \ell_\alpha^k|_i \ell_\beta^i = -\ell_j^\gamma|_i \ell_\alpha^j \ell_\beta^i. \end{cases} \tag{4.3}$$

If we have a tensor field of $(1,1)$-type for example (T_j^i), we denote by (T_β^α) the components of the tensor with respect to the Moór frame (ℓ_α^i), that is $T_\beta^\alpha = \ell_i^\alpha T_j^i \ell_\beta^j$. The scalar components of the metric tensor (g_{ij}) are $(\delta_{\alpha\beta})$, that is

$$g_{ij}\ell_\alpha^i \ell_\beta^j = \delta_{\alpha\beta}. \tag{4.4}$$

If $\mathcal{D}$ is the Cartan connection of the Finsler space, we know that with respect to it $\ell^i_{|j} = 0$, which is the same with $\ell_{i|j} = 0$. Then we find from (4.3) that the scalar components $F^\alpha_{1\beta}$ vanish.

Let us denote by "$|_\alpha$" and "$|_\alpha$" the scalar h-covariant and v-covariant derivative, respectively. They are given by

$$\begin{aligned} T^\alpha_{\beta|\gamma} &= H_\gamma(T^\alpha_\beta) + F^\alpha_{\delta\gamma}T^\delta_\beta - F^\delta_{\beta\gamma}T^\alpha_\delta, \quad \text{and} \\ T^\alpha_\beta|_\gamma &= V_\gamma(T^\alpha_\beta) + V^\alpha_{\delta\gamma}T^\delta_\beta - \nabla^\delta_{\beta\gamma}T^\alpha_\delta. \end{aligned} \tag{4.5}$$

The scalar covariant derivative and the covariant derivative are related by [2]

$$T^\alpha_{\beta|\gamma} = T^i_{j|k}\ell^\alpha_i\ell^j_\beta\ell^k_\gamma \quad \text{and} \quad T^\alpha_\beta|_\gamma = T^i_j|_k\ell^\alpha_i\ell^j_\beta\ell^k_\gamma. \tag{4.6}$$

As the fundamental tensor (g_{ij}) of a Finsler metric is h- and v-covariant constant, according to (4.4) and a corresponding formula for (4.5) we have that

$$\delta_{\alpha\beta|\gamma} = g_{ij|k}\ell^i_\alpha\ell^j_\beta\ell^k_\gamma = 0 \quad \text{and} \quad \delta_{\alpha\beta}|_\gamma = g_{ij}|_k\ell^i_\alpha\ell^j_\beta\ell^k_\gamma = 0.$$

From $\delta_{\alpha\beta|\gamma} = 0$ we have that $\delta_{\eta\beta}F^\eta_{\alpha\gamma}+\delta_{\alpha\eta}F^\eta_{\beta\gamma} = 0$, so $F^\alpha_{\beta\gamma}$ is skew symmetric with respect to α and β. As we already have that $F^\alpha_{1\beta} = 0$, then $F^1_{\alpha\beta} = 0$ and $F^\alpha_{\alpha\beta} = 0$. The nonzero scalar coefficients of the Cartan connection are $F^3_{2\gamma} = -F^2_{3\gamma}$. Denote by $h_j = F^3_{2\gamma}\ell^\gamma_j$. The Finsler vector field h_j is called the *h-connection vector* [8]. We denote by T, the scalar FF^3_{21}, so $T = F \cdot F^3_{21}$.

Proposition 4.2. *The dynamical covariant derivative of the Moór frame is given by*

$$\nabla\ell^i = 0, \quad \nabla m^i = Tn^i, \quad \textit{and} \quad \nabla n^i = -Tm^i. \tag{4.7}$$

Proof. The h-covariant derivative of the Moór frame with respect to Cartan connection is given by [8]

$$\ell^i_{|j} = 0, \quad m^i_{|j} = n^i h_j, \quad \text{and} \quad n^i_{|j} = -m^i h_j. \tag{4.8}$$

As $h_i y^i = F^3_{2\gamma}\ell^\gamma_i F \cdot \ell^i_1$; and $\ell^\gamma_j\ell^j_1 = \delta^\gamma_1$, we have that $h_j y^j = FF^3_{21} = T$ and (4.7) is true. □

Theorem 4.1. *A three dimensional Finsler space has scalar curvature if and only if*

$$B_{ij}m^i m^j = B_{ij}n^i n^j \quad \text{and} \quad B_{ij}m^i n^j = 0. \tag{4.9}$$

Proof. From (2.11) we can see that the Finsler space has scalar curvature if and only if $B_{ij} = RF^2 h_{ij}$. As B_{ij} is symmetric and $B_{ij}y^j = 0$, then with respect to Moór frame we have that

$$B_{ij} = Am_i m_j + B(m_i n_j + m_j n_i) + Cn_i n_j, \quad \text{where} \tag{4.10}$$

$$A = B_{ij}m^i m^j, \quad B = B_{ij}m^i n^j, \quad \text{and} \quad C = B_{ij}n^i n^j. \tag{4.11}$$

As $h_{ij} = m_i m_j + n_i n_j$, then a Finsler space has scalar curvature if $B_{ij} = RF^2(m_i m_j + n_i n_j)$. If we compare with (4.10) we have that $A = C = RF^2$ and $B = 0$, that is (4.9) is true. □

Conversely if (4.9) is true then, $B_{ij} = A(m_i m_j + n_i m_j) = Ah_{ij}$. So, the Finsler space has scalar curvature $R = \frac{1}{F^2}B_{ij}m^i m^j$.

Next we are looking for a scalar version (with respect to Moór frame) of the Jacobi equation (2.5). Consider $\big(X^i(x,y)\big)$ be a Finsler vector field with scalar components $\big(X^{(i)}(x,y)\big)$, that is: $X^i = X^{(1)}\ell^i + X^{(2)}m^i + X^{(3)}n^i$. According to Proposition 4.2 we have that the dynamical covariant derivative of (X^i) is given by

$$\nabla X^i = (X^{(1)})'\ell^i + [(X^{(2)})' - T]m^i + [(X^{(3)})' + T]n^i. \tag{4.12}$$

Then the second covariant derivative of (X^i) is given by

$$\begin{aligned}\nabla^2 X^i = (X^{(1)})''\ell^i &+ [(X^{(2)})'' - T' - T(X^{(3)})' - T^2]m^i \\ &+ [(X^{(3)})'' + T' + T(X^{(2)})' - T^2]n^i.\end{aligned} \tag{4.13}$$

Theorem 4.2. *The scalar form of the Jacobi equation* (2.5)*, in a three dimensional Finsler space, with respect to Moór frame is given by*

$$\begin{cases} (X^{(1)})'' = 0, \\ (X^{(2)})'' - T(X^{(3)})' + AX^{(2)} + BX^{(3)} - T' - T^2 = 0, \\ (X^{(3)})'' + T(X^{(2)})' + BX^{(2)} + CX^{(3)} + T' - T^2 = 0. \end{cases} \tag{4.14}$$

As for two dimensional Finsler spaces, we can see that the tangential component of a Jacobi vector field is a solution of $(4.14)_1$, that is $(at+b)\ell^i$.

If the scalar component F_{21}^3 of the Cartan connection vanishes, then $T = 0$ and the equations (4.14) become

$$\begin{cases} (X^{(1)})'' = 0, \\ (X^{(2)})'' + AX^{(2)} + BX^{(3)} = 0, \\ (X^{(3)})'' + BX^{(2)} + CX^{(3)} = 0. \end{cases} \tag{4.15}$$

If the Finsler space has scalar curvature R and $F_{21}^3 = 0$, then we have the following scalar version of the Jacobi equation (2.5):

$$\begin{cases} (X^{(1)})'' = 0, \\ (X^{(2)})'' + RF^2X^{(2)} = 0, \\ (X^{(3)})'' + RF^2X^{(3)} = 0. \end{cases} \tag{4.16}$$

In this case we may conclude like in two dimensional case that if the scalar curvature R is positive, then the geodesics are stable and if $R \leq 0$ then the geodesics are unstable.

Also we can use the equation (4.14) to determine the Lie symmetries of a three dimensional Finsler space.

5. Examples

(Finsler spaces applied to physics via computer algebra)

Examples of 2 dimensional spaces endowed with a Finsler (non-Riemannian) metric, stemming both from applications or pure mathematics, are not unusual in the literature. We will, therefore, concentrate on providing a higher dimensional example, worked out by means of Computer Algebra, derived from a non-Riemannian version of General Relativity.

A first explicit non-Riemannian solution to a generalized field equation [14] was produced by means of Computer Algebra [15] as the first order perturbation of the Riemannian 1 parameter family of metrics known as

the Schwarzschild solution (to Einstein's field equations):

$$ds^2 = \frac{dr^2}{(1-2m/r)} + r^2 d\Omega^2 - \left(1 - \frac{2m}{r}\right) dt^2 + \varepsilon \left(1 - \frac{2m}{r}\right) d\Omega dt, \quad (5.1)$$

where $d\Omega = \sqrt{d\theta^2 + \sin^2\theta d\phi^2}$ and the perturbation parameter ε is considered small ($\varepsilon^2 \approx 0$). This solution has been obtained by S. F. Rutz in 1992 [16].

Both Riemannian ($\varepsilon = 0$) and non-Riemannian ($\varepsilon \neq 0$) families of metrics are solutions to vacuum field equations, valid outside matter, and are invariant under SO(3), leading to the so-called spherical symmetry [17], proper to model physical systems such as space-time in the vicinity of massive stars or black holes.

The effect of the parameter m, which stands for the mass of the star or black hole and relates to the curvature of the Riemannian manifold, is well-known. In order to determine the contribution of the non-Riemannian term in (5.1), let us take $m = 0$. In the Riemannian case ($\varepsilon = 0$), this leads directly to the so-called Minkowski metric, a strictly flat space, with straight lines as geodesics, a classical model for physical space-time, as described in Special Relativity. As it is well-known, geodesics represent the trajectories of test particles or light rays under the action of the gravitational field produced by the mass m, is this case, in empty space. But the notion of straight lines as trajectories of free particles precedes Newton, dating back to Galileo. Nevertheless, just by allowing for non-Riemannian models of space-time, one arrives at

$$ds^2 = dr^2 + r^2 d\Omega^2 - dt^2 + \varepsilon d\Omega dt, \quad (5.2)$$

as a possible description for empty space. In what such model differs from the classical view? To try to answer this, let us now make use of the Computer Algebra package Finsler [18] to determine the expressions of some Finslerian tensors taking (5.2) as input, and considering ($\varepsilon^2 \approx 0$).

5.1. Spray theoretical results. As mentioned before, (5.2) solves a generalized vacuum field equation, this resulting from the fact that Berwald's deviation tensor is identically zero up to order ε,

$$B^i{}_j = \mathrm{O}(\varepsilon^2).$$

From this we have also that Berwald's curvature tensor $R^i_{jkl} = \mathrm{O}(\varepsilon^2)$. Most components of Douglas' tensor G^i_{jkl}([1]) are also zero up to order ε, the non-zero ones being([2])([3])

$$G^t{}_{r\theta\theta} = \frac{1}{2}\frac{\sin^2\theta\, d\phi^2}{d\Omega^3\, r}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$G^t{}_{r\theta\phi} = -\frac{1}{2}\,\frac{d\theta\, \sin^2\theta\, d\phi}{d\Omega^3\, r}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$G^t{}_{r\phi\phi} = \frac{1}{2}\,\frac{\sin^2\theta\, d\theta^2}{d\Omega^3\, r}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$G^t{}_{\theta\theta\theta} = -\frac{3}{2}\,\frac{dr\, \sin^2\theta\, d\phi^2\, d\theta}{d\Omega^5\, r}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$G^t{}_{\theta\theta\phi} = \frac{1}{2}\,\frac{dr\, \sin^2\theta\, d\phi\, (2\, d\theta^2 - \sin^2\theta\, d\phi^2)}{d\Omega^5\, r}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$G^t{}_{\theta\phi\phi} = -\frac{1}{2}\,\frac{d\theta\, dr\, \sin^2\theta\, (-2\, \sin^2(\theta)\, d\phi^2 + d\theta^2)}{d\Omega^5\, r}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$G^t{}_{\phi\phi\phi} = -\frac{3}{2}\,\frac{\sin^4\theta\, dr\, d\theta^2\, d\phi}{d\Omega^5\, r}\,\varepsilon + \mathrm{O}(\varepsilon^2).$$

Note that every non-zero component of G^i_{jkl} is of the form G^t_{jkl}, where none of the lower indexes is t, which implies that we have Riemann-like (Berwald) geodesic equations for $x^i \neq t$, and the correction to the t-geodesic equation is independent of dt. Geodesics for (5.1) have been explicitly worked out in [14].

This result shows that such spaces are *non Berwald*. It also means they are *not projectively flat* as well. From Berwald's deviation tensor we have that the Flag curvature K is also zero up to order ε,

$$K = \mathrm{O}(\varepsilon^2),$$

[1]We have the symbol G^i_{jkl} for Douglas' tensor in the computer package due to the special nature of the symbol D, usually reserved for differentiation in Computer Algebra.

[2]The components that may be obtained by the symmetry properties of the tensor in questions are not shown.

[3]The following computer generated expressions have been modified in order to be made more compact. Particularly in the denominators, the expression $\sqrt{d\theta^2 + \sin(\theta)^2 d\phi^2}$ has been substituted for $d\Omega$.

which means that the space is of *constant curvature*, which implies that it is of *scalar curvature*, which in turn, by Szabó theorem [3], implies that Weyl's projective tensor is also zero up to order ε, $W^i_{jkl} = \mathrm{O}(\varepsilon^2)$. But, as Douglas' projective tensor $\Pi^i_{jkl} = \dot\partial_j\dot\partial_k\dot\partial_l(G^i - (1/5)\dot\partial_a G^a\dot x^i)$ is not zero up to order ε, the non-zero components being

$$\Pi^t{}_{r\theta\theta} = \frac{1}{2}\,\frac{\sin^2\theta\,d\phi^2}{d\Omega^3\,r}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$\Pi^t{}_{r\theta\phi} = -\frac{1}{2}\,\frac{d\theta\,\sin^2\theta\,d\phi}{d\Omega^3\,r}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$\Pi^t{}_{r\phi\phi} = \frac{1}{2}\,\frac{\sin^2\theta\,d\theta^2}{d\Omega^3\,r}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$\Pi^t{}_{\theta\theta\theta} = -\frac{3}{2}\,\frac{dr\,\sin^2\theta\,d\phi^2\,d\theta}{d\Omega^5\,r}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$\Pi^t{}_{\theta\theta\phi} = -\frac{1}{2}\,\frac{dr\,\sin^2\theta\,d\phi\,(-2\,d\theta^2 + \sin^2\theta\,d\phi^2)}{d\Omega^5\,r}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$\Pi^t{}_{\theta\phi\phi} = -\frac{1}{2}\,\frac{d\theta\,dr\,\sin^2\theta\,(-2\,\sin^2\theta\,d\phi^2 + d\theta^2)}{d\Omega^5\,r}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$\Pi^t{}_{\phi\phi\phi} = -\frac{3}{2}\,\frac{\sin^4\theta\,dr\,d\theta^2\,d\phi}{d\Omega^5\,r}\,\varepsilon + \mathrm{O}(\varepsilon^2),$$

which implies that these spaces are *not projectively Berwald*, or not projectively equivalent to a Berwald space.

Finally, the non-zero components of $D^0_{jkl} = g_{im}\dot x^m G^i_{jkl}$, are

$$D^0{}_{r\theta\theta} = -\frac{1}{2}\,\frac{dt\,\sin^2\theta\,d\phi^2}{d\Omega^3\,r}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$D^0{}_{r\theta\phi} = \frac{1}{2}\,\frac{dt\,d\theta\,\sin^2\theta\,d\phi}{d\Omega^3\,r}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$D^0{}_{r\phi\phi} = -\frac{1}{2}\,\frac{dt\,\sin^2\theta\,d\theta^2}{d\Omega^3\,r}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$D^0{}_{\theta\theta\theta} = \frac{3}{2}\,\frac{dt\,dr\,\sin^2\theta\,d\phi^2\,d\theta}{d\Omega^5\,r}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$D^0{}_{\theta\theta\phi} = \frac{1}{2}\,\frac{dt\;dr\;\sin^2\theta\,d\phi\,(-2\,d\theta^2+\sin^2\theta\,d\phi^2)}{d\Omega^5\,r}\,\varepsilon+\mathrm{O}(\varepsilon^2)$$

$$D^0{}_{\theta\phi\phi} = \frac{1}{2}\,\frac{dt\,d\theta\;dr\;\sin^2\theta\,(-2\,\sin^2\theta\,d\phi^2+d\theta^2)}{d\Omega^5\,r}\,\varepsilon+\mathrm{O}(\varepsilon^2)$$

$$D^0{}_{\phi\phi\phi} = \frac{3}{2}\,\frac{dt\;\sin^4\theta\,dr\,d\theta^2\,d\phi}{d\Omega^5\,r}\,\varepsilon+\mathrm{O}(\varepsilon^2)$$

what tells us that such spaces are *non Landsberg*.

So, in a spray-like perspective, we have that the geodesics of (5.2) are corrected only for the t coordinate, that such correction is independent of dt, and that the geodesics deviate linearly from one another. In particular this last feature connects our model to the classical intuition about trajectories in empty space-time.

We say (5.2) is an *almost flat* space, since $R^i_{jkl}=0$ and most components of D^i_{jkl} are also zero, in the given order of approximation.

5.2. Cartan-type metrical results. In a metrical perspective, using Cartan's formalism for Finsler spaces, we have, to begin with, that Cartan's tensor C_{ijk} is not only nonzero, but actually proportional to the perturbation parameter ε,

$$C_{\theta\theta\theta} = -\frac{3}{2}\,\frac{dt\;\sin^2\theta\,d\phi^2\,d\theta}{d\Omega^5}\,\varepsilon$$

$$C_{\theta\theta\phi} = \frac{1}{2}\,\frac{dt\;\sin^2\theta\,d\phi\,(2\,d\theta^2-\sin^2\theta\,d\phi^2)}{d\Omega^5}\,\varepsilon$$

$$C_{\theta\theta t} = \frac{1}{2}\,\frac{\sin^2\theta\,d\phi^2}{d\Omega^3}\,\varepsilon$$

$$C_{\theta\phi\phi} = -\frac{1}{2}\,\frac{dt\,d\theta\;\sin^2\theta\,(-2\,\sin^2\theta\,d\phi^2+d\theta^2)}{d\Omega^5}\,\varepsilon$$

$$C_{\theta\phi t} = -\frac{1}{2}\,\frac{d\theta\;\sin^2\theta\,d\phi}{d\Omega^3}\,\varepsilon$$

$$C_{\phi\phi\phi} = -\frac{3}{2}\,\frac{dt\;\sin^4\theta\,d\theta^2\,d\phi}{d\Omega^5}\,\varepsilon$$

$$C_{\phi\phi t} = \frac{1}{2}\,\frac{\sin^2\theta\;dth^2}{d\Omega^3}\,\varepsilon.$$

This tells us that spaces endowed with (5.2) as metric function are *not Riemannian*. Actually, such spaces would be Riemannian if and only if $\varepsilon = 0$, when would have a Minkowski metric (in the relativistic sense). The same is true for (5.1), which is Riemannian if and only if $\varepsilon = 0$, when we would have the Schwarzschild metric.

The fact that spaces with a metric as (5.2) are non-Landsberg, as stated before, gives us that Cartan's curvature tensor P^i_{jkl} is not zero up to order ε. But we have that Cartan's curvature tensor S^i_{jkl} is *identically zero* up to the same order,

$$S_{ijkl} = \mathrm{O}(\varepsilon^2).$$

This result implies that the curvature of the tangent spaces to (5.2) is zero, that is, given a fixed point in the manifold one could find y^i-coordinates such that the tangent space at that point is Euclidean, as in Riemannian spaces. Note that such y^i-transformations are not allowed in Finsler spaces independently of its correspondent x^i-transformations, such being considered as an illustration here.

Note also that Brickell's theorem [5], that says that $S_{ijkl} = 0$ implies that the space must be Riemannian, does not apply here, since (5.2) is not positive-definite. The metric (5.2) thus provide an example of how Brickell's theorem fails if its assumptions, particularly regarding positive-definiteness, are not met.

As for the third Cartan's curvature tensor, R^i_{jkl}, given as RC^i_{jkl} in the output below to differ from Berwald's curvature tensor R^i_{jkl}, is not zero up to order ε. For instance, the component

$$RC^r{}_{\theta r\theta} = \frac{1}{2}\,\frac{dt\,\sin^2\theta\,d\phi^2}{d\Omega^3\,r^2}\,\varepsilon + \mathrm{O}(\varepsilon^2).$$

This shows that, although (5.2) is a first order departure of a Riemannian flat space, we have that only 1 among the 3 Cartan's curvature tensors, namely S_{ijkl}, is zero in the same order of approximation.

The full description of the metrical properties of (5.2) will be given elsewhere [20].

5.3. Frame fields. In order to express (5.2) in terms of frame fields, or tetrads, as such fields are known in Relativity, let us now first look at the 3

dimensional metric produced by taking the coordinate r to be constant, or $dr = 0$, thereby still preserving the non-Riemannian character in

$$ds_3^2 = r^2 d\Omega^2 - dt^2 + \varepsilon d\Omega dt. \tag{5.3}$$

The Moór frame for such 3 dimensional metric is given, up to first order of approximation in ε, by

$$n1^{\theta} = \frac{d\theta}{(r^2 d\Omega^2 - dt^2)^{(1/2)}} - \frac{1}{2}\,\frac{d\theta\, d\Omega\, dt}{(r^2 d\Omega^2 - dt^2)^{(3/2)}}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$n1^{\phi} = \frac{d\phi}{(r^2 d\Omega^2 - dt^2)^{(1/2)}} - \frac{1}{2}\,\frac{d\phi\, d\Omega\, dt}{(r^2 d\Omega^2 - dt^2)^{(3/2)}}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$n1^{t} = \frac{dt}{(r^2 d\Omega^2 - dt^2)^{(1/2)}} - \frac{1}{2}\,\frac{dt^2\, d\Omega}{(r^2 d\Omega^2 - dt^2)^{(3/2)}}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$n2^{\theta} = -\frac{d\theta\, dt\, d\Omega}{r\,\sqrt{-r^2 d\Omega^2 + dt^2}\,(-d\Omega^2)} - \frac{1}{2}\,\frac{d\theta\, r\,(-d\Omega^2)}{(-r^2 d\Omega^2 + dt^2)^{(3/2)}}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$n2^{\phi} = -\frac{d\phi\, dt\, d\Omega}{r\,\sqrt{-r^2 d\Omega^2 + dt^2}\,(-d\Omega^2)} - \frac{1}{2}\,\frac{d\phi\, r\,(-d\Omega^2)}{(-r^2 d\Omega^2 + dt^2)^{(3/2)}}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$n2^{t} = -\frac{d\Omega r}{r\,\sqrt{-r^2 d\Omega^2 + dt^2}\,(-d\Omega^2)} + \frac{1}{2}\,\frac{dt^3}{(-r^2 d\Omega^2 + dt^2)^{(3/2)}}\,\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$n3^{\theta} = -\frac{d\theta\, dt}{r\, d\Omega\,\sqrt{-r^2 d\Omega^2 + dt^2}} + \frac{1}{2}\frac{\sqrt{-r^2 d\Omega^2 + dt^2}\,(-d\Omega)\, d\theta\, r}{(r^2 d\Omega^2 - dt^2)^2}\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$n3^{\phi} = -\frac{d\phi\, dt}{r\, d\Omega\,\sqrt{-r^2 d\Omega^2 + dt^2}} + \frac{1}{2}\frac{\sqrt{-r^2 d\Omega^2 + dt^2}\,(-d\Omega)\, d\phi\, r}{(r^2 d\Omega^2 - dt^2)^2}\varepsilon + \mathrm{O}(\varepsilon^2)$$

$$n3^{t} = -\frac{-d\Omega^2\, r}{d\Omega\,\sqrt{-r^2 d\Omega^2 + dt^2}} - \frac{1}{2}\frac{\sqrt{-r^2 d\Omega^2 + dt^2}\; - dt^3}{r(r^2 d\Omega^2 - dt^2)^2}\varepsilon + \mathrm{O}(\varepsilon^2).$$

The frame above reproduce the metric (5.3) in the given order of approximation, as

$$ds_3^2 = (n1_i n1_i + n2_i n2_j + n3_i n3_j) dx^i dx^j,$$

with $dx^i = d\theta, d\phi, dt$. In order to produce the actual frame for the 4 dimensional metric (5.2), we may use the Collaring theorem from Differential Topology to claim that there exists a vector field in the 4 dimensional

Finsler space given by (5.2) which is normal to its subspace $dr = 0$, given by (5.3). It is important to note that the fact that a Finsler metric like (5.2) can be expressed in terms of a frame field, thus pointing out to the generalization of the powerful calculus in tetrad fields from its usual Riemannian framework in General Relativity to Finsler spaces, is of central importance to the development of generalized theories of gravity, that may allow for non straight behavior of particle trajectories in empty space-time, such as has been recently suggested by deep space observations.

As further steps, we want to determine such a frame field, and also to similarly work out the (tetrad) frame field for the 4 dimensional metric (5.1), and therefore proceed to determine an axially symmetric non-Riemannian solution to the generalized gravitational field equation, in the line described in [19].

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Appendix

A1. Consider F the fundamental function of a Finsler space.

(1) $g_{ij} := \frac{1}{2}\,\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is the metric tensor, (g^{ij}) the inverse;

(2) $G^i := \frac{1}{4}\,g^{ij}\Big(\frac{\partial^2 F^2}{\partial y^j \partial x^m} y^m - \frac{\partial F^2}{\partial x^j}\Big)$ are the local coefficients of the canonical spray;

(3) $N^i_j := \frac{\partial G^i}{\partial y^j}$ are the local coefficients of the canonical nonlinear connection;

(4) $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i\,\frac{\partial}{\partial y^h}$ is the h-operator of differentiation;

(5) $R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}$ is the curvature of the nonlinear connection (if $R^i_{jk} = 0$ the nonlinear connection is integrable);

(6) $B^i_j := R^i_{jk} y^k$ is the second invariant in KCC-theory;
$B_{ij} := g_{ik} B^k_j$ (it is symmetric, its first eigenvalue is always zero);

(7) $h_{ij} = F\,\frac{\partial^2 F}{\partial y^i \partial y^j}$ is the angular metric (determinant of h_{ij} is 0) (if B_{ij} and h_{ij} are proportional, then the Finsler space has scalar curvature $R = \frac{1}{F^2}\,\frac{B_{ij}}{h_{ij}}$);

(8) $C_{ijk} = \frac{1}{2}\,\frac{\partial g_{ij}}{\partial y^k}$ and $C^i_{jk} = g^{i\ell} C_{\ell jk}$ are Cartan tensors;

(9) $C^i = C^i_{jk} g^{jk}$ is the Cartan vector;

(10) $C = (C^i C^j g_{ij})^{1/2}$ is the length of the Cartan vector;

(11) $\ell_i = \frac{1}{F}\,y_i$ is the normalized supporting element; (we have that $g_{ij} = h_{ij} + \ell_i \ell_j$)

A2. Two dimensional Finsler space.

(12) $m^i = \frac{1}{C}\,C^i$; ℓ^i and m^i are orthogonal, that is, $g_{ij}\ell^i m^j = 0$ $\quad(\ell^i, m^i)$ is called the Berwald frame;

(13) $F^i_{jk} = -\ell_j\,\frac{\delta \ell^i}{\delta x^k} - m_j\,\frac{\delta m^i}{x^k}$ are the horizontal coefficients of the Cartan connections (are symmetric): $F^i_{jk} = F^i_{kj}$ and $\ell_i = g_{ij}\ell^j$ and $m_i = g_{ij} m^j$;

(14) $C^i_{jk} = -\ell_j\,\frac{\partial \ell^i}{\partial y^k} - m_j\,\frac{\partial m^i}{\partial y^k} + \frac{1}{F}\,\ell_j m^i m_k - \frac{1}{F}\,m_j \ell^i m_k$ are the vertical components of the Cartan connection.

(15) Here we have that $B_{ij} = RF^2 h_{ij} = RF^2 m_i m_j$, evaluate R. If a vector field (X^i) has the components $X^{(1)}$ and $X^{(2)}$ with respect to Berwald frame then $\begin{cases} (X^{(1)})' = 0, \\ (X^{(2)})'' + RF^2 X^{(2)} = 0. \end{cases}$

A3. Three dimensional Finsler spaces.

ℓ^i and m^i are from (11) and (12).

(16) $g = \det(g_{ij})$

(17) $\begin{cases} n^1 = g^{-1/2}(\ell_2 m_3 - \ell_3 m_2) \\ n^2 = g^{-1/2}(\ell_3 m_1 - \ell_1 m_3) \\ n^3 = g^{-1/2}(\ell_1 m_2 - \ell_2 m_1) \end{cases}$

(n^i) is a unitary vector field: $g_{ij} n^i n^j = 1$ (ℓ^i, m^i, n^i) is an orthogonal base, called the Moór base.

(18) $\begin{cases} A = B_{ij} m^i m^j \\ B = B_{ij} m^i n^j \\ C = B_{jij} n^i n^j \end{cases}$

if $A = C$ and $B = 0$ the space has scalar curvature $R = \frac{1}{F^2} A$.

(19) $T := \left(\frac{\partial m^1}{\partial x^j} y^j - 2 \frac{\partial m^1}{\partial y^j} G^j \right) / n^1$ ($\frac{1}{F} T = F^3_{21}$ a scalar component of the Cartan connection).

(20) If $T = 0$ then the scalar components $(X^{(i)})$, with respect to Moór frame, of a Jacobi vector field are solutions of

$$\begin{cases} (X^{(1)})'' = 0, \\ (X^{(2)})'' + AX^{(2)} + BX^{(3)} = 0, \\ (X^{(3)})'' + BX^{(2)} + CX^{(3)} = 0. \end{cases}$$

(21) If $T \neq 0$ the scalar components of a Jacobi vector field are given by the system

$$\begin{cases} (X^{(1)})'' = 0, \\ (X^{(2)})'' - T(X^{(3)})' + AX^{(2)} + BX^{(3)} - T' - T^2 = 0, \\ (X^{(3)})'' + T(X^{(2)})' + BX^{(2)} + CX^{(3)} + T' - T^2 = 0. \end{cases}$$

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