

On functional equations involving means

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Dedicated to the 80th birthday of Professor Lajos Tamássy

Abstract. The main results of the paper concern functional equations of the form

$$f(M(x, y))(g(y) - g(x)) = \mu(f(x)g(y) - f(y)g(x)) \quad (x, y \in I),$$

where f and g are continuous functions defined on an open interval I and M is a strict two variable mean on I . As an application, a generalization of the so-called Matkowski–Sutô problem for weighted two variable quasi-arithmetic means is solved under first-order continuous differentiability assumptions.

1. Introduction

Let $I \subset \mathbb{R}$ be a nonvoid open interval. We say that a function $M : I^2 \rightarrow I$ is a *pre-mean* on I if

$$\min\{x, y\} \leq M(x, y) \leq \max\{x, y\} \tag{1}$$

Mathematics Subject Classification: 39B22, 39B12, 26A18.

Key words and phrases: functional equation, weighted quasi-arithmetic mean, Stolarsky means, Matkowski–Sutô problem.

This research has been supported by the Hungarian National Research Science Foundation (OTKA) Grant T-0300082 and by the Higher Education, Research, and Development Fund (FKFP) Grant 0310/1997.

for all $x, y \in I$. If M is a pre-mean on I and the inequalities in (1) are strict for all $x \neq y$ ($x, y \in I$) then M is called a *strict pre-mean* on I . If M is a pre-mean on I and continuous on I^2 then M is said to be a *mean* on I . Finally, M is called a *strict mean* on I if M is a strict pre-mean on I and continuous on I^2 .

In this paper we deal with functional equations of the form

$$f(M(x, y)) = N(f(x), f(y)) \quad \text{if } f(x) \neq f(y) \quad (x, y \in I) \quad (2)$$

and

$$f(M(x, y))(g(y) - g(x)) = \mu(f(x)g(y) - f(y)g(x)) \quad (x, y \in I) \quad (3)$$

under the following conditions:

(A) In (2) M is a strict mean on I , N is strict pre-mean on J ($\subset \mathbb{R}$ is a nonvoid open interval) and the unknown function $f : I \rightarrow J$ is continuous;

(B) In (3) M is a mean (or a strict mean) on I and the unknown functions $f, g : I \rightarrow \mathbb{R}_+$ are continuous (here and in the sequel $\mathbb{R}_+$ denotes the set of positive real numbers).

As an application, we shall consider the Matkowski–Sutô type problem

$$\lambda x + (1 - \lambda)y = \mu M(x, y) + (1 - \mu)N(x, y) \quad (x, y \in I) \quad (4)$$

where $0 < \lambda < 1$, $\mu \neq 0, 1$ are constants, M and N are weighted quasi-arithmetic means on I with weight λ . The case $\lambda = \mu = \frac{1}{2}$ is the original Matkowski–Sutô problem (cf. [1], [2], [4], [3], [7], [13], [14]). In [5], the case $\lambda = \mu$ was treated. The aim of this paper is to solve the above problem if the *generator functions* of M and N are continuously differentiable with nonvanishing derivatives on I . Our results offer the same set of solutions that was found by GŁAZOWSKA, JARCZYK, and MATKOWSKI [8] under twice continuous differentiability assumptions in the case $\lambda = 1/2$.

2. Injectivity Theorem

The following result, concerning the functional equation (2) is an injectivity theorem of independent interest.

Theorem 1. *Let $I, J \subset \mathbb{R}$ be nonvoid open intervals and let a strict mean $M : I^2 \rightarrow I$ and a strict pre-mean $N : J^2 \rightarrow J$ be given. If $f : I \rightarrow J$ is a continuous and nonconstant solution of the (conditional) functional equation*

$$f(M(x, y)) = N(f(x), f(y)) \quad \text{if } f(x) \neq f(y) \quad (x, y \in I) \quad (2)$$

then f is injective on I .

PROOF. Suppose that $f : I \rightarrow J$ is a *continuous* solution of (2) which is *nonconstant* on I and *not injective*. Since f is continuous, there exist $a < b$ ($a, b \in I$) such that $f(a) = f(b) =: k \in J$ and $f(x) \neq k$ if $x \in]a, b[$. We show that there *exist* numbers $A, B \in [a, b]$ such that

$$M(a, B) = A \quad \text{and} \quad M(A, b) = B \quad (5)$$

hold. For, consider the function

$$\varphi(x, y) := (M(a, y), M(x, b)) \quad (x, y \in [a, b]).$$

The function M being continuous, $\varphi : [a, b]^2 \rightarrow [a, b]^2$ is a continuous self-mapping of $[a, b]^2$, hence, by Brouwer's Fixed Point Theorem, it has a fixed point $(A, B) \in [a, b]^2$, that is,

$$\varphi(A, B) = (A, B).$$

This means that the equations (5) hold. Since M is a strict mean, we necessarily have

$$a < A < B < b.$$

This implies $k = f(a) \neq f(B)$ and $k = f(b) \neq f(A)$, and by (5) and (2),

$$f(A) = f(M(a, B)) = N(f(a), f(B)) = N(k, f(B))$$

and

$$f(B) = f(M(A, b)) = N(f(A), f(b)) = N(f(A), k).$$

Since N is a strict pre-mean, the previous equations yield that $f(A)$ belongs to the open interval joining k and $f(B)$, and similarly, $f(B)$ belongs to the open interval joining $f(A)$ and k . Since $f(x) \neq k$ if $x \in]a, b[$, we have reached a contradiction. Therefore, if f is a continuous and nonconstant solution of (2) then f is injective on I . $\square$

3. Equation (3)

The following result is of basic importance.

Theorem 2. *Let $M : I^2 \rightarrow I$ be a mean on I and let $\mu \neq 0, 1$ be a real constant. If $f, g : I \rightarrow \mathbb{R}_+$ are continuous solutions of the functional equation*

$$f(M(x, y))(g(y) - g(x)) = \mu(f(x)g(y) - f(y)g(x)) \quad (x, y \in I) \quad (3)$$

then there exists $c \in \mathbb{R}_+$ such that

$$f(x)^\mu g(x)^{1-\mu} = c \quad (6)$$

for all $x \in I$.

PROOF. If $g(x) = g(y)$ (> 0) then (3) implies $f(x) = f(y)$, thus there exists a function $F : g(I) \rightarrow \mathbb{R}_+$ such that

$$f(x) = F(g(x)) \quad (x \in I). \quad (7)$$

If g is constant on I then f is constant on I , too, and clearly (6) holds. Therefore we can assume that g is *nonconstant* on I . Since g is continuous, then $J := g(I)$ is a proper interval and $J \subset \mathbb{R}_+$.

In what follows, we intend to show that the function

$$F : J \rightarrow \mathbb{R}_+$$

is *differentiable* on J .

Let $u \in J$ and let $\{u_n \mid n \in \mathbb{N}\} \subset J$ be a sequence such that $u_n \rightarrow u$ from the left ($u_n < u$) (or from the right $u_n > u$). It is sufficient to show that

$$\frac{F(u_n) - F(u)}{u_n - u}$$

tends to the same limit ($n \rightarrow \infty$) depending only on u .

Let

$$u_0 := \inf\{u_n \mid n \in \mathbb{N}\} = \min\{u_n \mid n \in \mathbb{N}\}.$$

Then there exist $x_0, x^* \in I$ such that $g(x_0) = u_0$ and $g(x^*) = u$. We may assume that $x_0 < x^*$, the other case can be handled similarly.

Let

$$H := \{t \in I \mid x_0 \leq t \leq x^* \text{ and } g(t) = u\}.$$

Then H is a *closed* set, and since $x^* \in H$, H is *not empty*. Denote

$$x := \inf H,$$

then we clearly have that $x_0 < x$. Since g is continuous, $g(x) = u$ and if $x_0 \leq t < x$ then $g(t) \neq u$. The function g takes each value between u_0 and u on the closed interval $[x_0, x]$, therefore there *exists* a sequence $x_n \in [x_0, x]$ such that $g(x_n) = u_n$ ($n \in \mathbb{N}$). We show that $x_n \rightarrow x$ as $n \rightarrow \infty$. If this were not so then there would be a subsequence (x_{n_k}) ($n_1 < n_2 < n_3 < \dots$) converging to $\bar{x} \neq x$, and from this we would have $\bar{x} < x$. Since g is continuous, $g(x_{n_k}) \rightarrow g(\bar{x})$ ($k \rightarrow \infty$) and $g(x_{n_k}) = u_{n_k} \rightarrow u = g(x)$ ($k \rightarrow \infty$). These would imply $g(\bar{x}) = g(x)$, contradicting the definition of x . Therefore $x_n \rightarrow x$ as $n \rightarrow \infty$ indeed.

The previous and equation (3) imply

$$\begin{aligned} f(M(x_n, x)) &= \mu \frac{f(x_n)g(x) - f(x)g(x_n)}{g(x) - g(x_n)} \\ &= \mu \frac{F(u_n)u - F(u)u_n}{u - u_n} = \mu \left(-u \frac{F(u_n) - F(u)}{u_n - u} + F(u) \right). \end{aligned}$$

Since f is continuous and M is a mean, we have that

$$\lim_{n \rightarrow \infty} f(M(x_n, x)) = f(x) = F(g(x)) = F(u).$$

Thus, by the previous equation, the limit

$$\lim_{n \rightarrow \infty} \frac{F(u_n) - F(u)}{u_n - u} = F'(u)$$

exists, too, and

$$F(u) = \mu(-uF'(u) + F(u))$$

holds for all $u \in J$. Since $\mu \neq 0, 1$, from this we obtain

$$\left(\ln F(u) - \frac{\mu - 1}{\mu} \ln u \right)' = 0$$

for all $u \in J$, that is, there exists a constant $q > 0$ satisfying

$$F(u) = qu^{\frac{\mu-1}{\mu}} \quad (u \in J).$$

By (7), this implies the assertion of the theorem. $\square$

In view of Theorem 2, the investigation of the functional equation (3) reduces to the discussion of a functional equation involving only one unknown function. The following results deal with this reduced problem.

Corollary 1. *Let $M : I^2 \rightarrow I$ be a mean on I and let $\mu \neq 0, 1$ be a real constant. If $f, g : I \rightarrow \mathbb{R}_+$ are continuous solutions of the functional equation (3) then $f : I \rightarrow \mathbb{R}_+$ is a continuous solution of the functional equation*

$$f(M(x, y)) \left(f(x)^{\frac{\mu}{1-\mu}} - f(y)^{\frac{\mu}{1-\mu}} \right) = \mu \left(f(x)^{\frac{\mu}{1-\mu}+1} - f(y)^{\frac{\mu}{1-\mu}+1} \right) \quad (8)$$

$$(x, y \in I).$$

Theorem 3. *Let $M : I^2 \rightarrow I$ be a strict mean and $\mu \neq 0, 1$. If $f : I \rightarrow \mathbb{R}_+$ is a continuous solution of the functional equation (8) then either there exists $c > 0$ such that*

$$f(x) = c \quad \text{if } x \in I, \quad (9)$$

or, f is injective on I and

$$f(M(f^{-1}(u), f^{-1}(v))) = S_{\frac{\mu}{1-\mu}+1, \frac{\mu}{1-\mu}}(u, v) \quad (10)$$

for all $u \neq v$, $u, v \in f(I) =: J \subset \mathbb{R}_+$. Here $S_{a,b} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is the Stolarsky mean on $\mathbb{R}_+$ of parameters a, b defined in the case $ab(a-b) \neq 0$ by

$$S_{a,b}(u, v) := \begin{cases} \left(\frac{b(u^a - v^a)}{a(u^b - v^b)} \right)^{\frac{1}{a-b}} & \text{if } u \neq v \\ u & \text{if } u = v \end{cases} \quad (u, v \in \mathbb{R}_+). \quad (11)$$

(cf. [11], [12]).

PROOF. Under these conditions (9) always solves (8), thus we can suppose that f is a nonconstant continuous solution of (8). Then if $f(x) \neq$

$f(y)$ $x, y \in I$, (8) yields

$$f(M(x, y)) = \mu \frac{f(x)^{\frac{\mu}{1-\mu}+1} - f(y)^{\frac{\mu}{1-\mu}+1}}{f(x)^{\frac{\mu}{1-\mu}} - f(y)^{\frac{\mu}{1-\mu}}} = S_{\frac{\mu}{1-\mu}+1, \frac{\mu}{1-\mu}}(f(x), f(y)). \quad (12)$$

Since the Stolarsky means defined by (11) are strict, it follows from Theorem 1 that f is injective. Thus, for all $u, v \in f(I) =: J$ with $u \neq v$, (12) implies (10). $\square$

In the sequel we shall need the following lemma.

Lemma 1. *Let $K \subset \mathbb{R}_+$ be a nonvoid open interval. The Stolarsky mean $S_{a,b}$ of parameters a, b ($ab(a-b) \neq 0$) is a quasi-arithmetic mean on K if and only if either $a = 2b$ or $2a = b$ or $a + b = 0$.*

PROOF. Let φ be a continuous strictly monotonic function such that

$$S_{a,b}(u, v) = \varphi^{-1} \left(\frac{\varphi(u) + \varphi(v)}{2} \right) =: A_{\varphi}(u, v) \quad (u, v \in K).$$

Then for all $t > 0$ and $u, v \in K$ with $tu, tv \in K$, $S_{a,b}(tu, tv) = tS_{a,b}(u, v)$ holds, which yields

$$A_{\varphi}(tu, tv) = tA_{\varphi}(u, v).$$

Thus, the quasi-arithmetic mean A_{φ} is homogeneous on the interval K . Now, by [10, Corollary 2],

$$A_{\varphi}(u, v) = H_p(u, v) \quad (u, v \in K)$$

for some parameter p , where

$$H_p(u, v) := \begin{cases} \left(\frac{u^p + v^p}{2} \right)^{\frac{1}{p}} & \text{if } p \neq 0 \\ \sqrt{uv} & \text{if } p = 0 \end{cases} \quad (u, v \in K)$$

is the p th Hölder (or power) mean. On the other hand, by the comparison theorem of Stolarsky means (see [9] and also [6]) $S_{a,b} = H_p = S_{2p,p}$ if and only if $a = 2b$ or $2a = b$ or $a + b = 0$. $\square$

4. The generalization of the Matkowski–Sutô problem

Let $\mathcal{CM}(I)$ denote the class of strictly monotonic and continuous functions defined on I . A mean $M : I^2 \rightarrow I$ is called a *weighted quasi-arithmetic mean* on I if there exist a constant $0 < \lambda < 1$ and $\varphi \in \mathcal{CM}(I)$ such that

$$M(x, y) = \varphi^{-1}(\lambda\varphi(x) + (1 - \lambda)\varphi(y)) =: A_\varphi(x, y; \lambda) \quad (13)$$

for all $x, y \in I$. If $\varphi(x) = \text{id}(x) := x$ then we write

$$A_\varphi(x, y; \lambda) = A(x, y; \lambda) = \lambda x + (1 - \lambda)y \quad (x, y \in I).$$

The number $0 < \lambda < 1$ is called a *weight* and the function $\varphi \in \mathcal{CM}(I)$ is called a *generator function*.

A possible generalization of the Matkowski–Sutô problem can be formulated in the following way: *Determine the constants $0 < \lambda < 1$ and $\mu \neq 0, 1$ and the functions $\varphi, \psi \in \mathcal{CM}(I)$ such that*

$$\lambda x + (1 - \lambda)y = \mu A_\varphi(x, y; \lambda) + (1 - \mu)A_\psi(x, y; \lambda) \quad (14)$$

holds for all $x, y \in I$. The case $\lambda = \mu = \frac{1}{2}$ is the original Matkowski–Sutô problem whose solution with analyticity assumptions was determined to SUTÔ [13], [14]. MATKOWSKI [7] found the same set of solutions under twice continuous differentiability assumptions. The unnatural regularity assumptions were step by step eliminated in a sequence of papers [1], [2], [4] by the authors and GY. MAKSA. Finally, the complete solution was found in [3].

Now we consider the equation (14) under the following assumptions: φ and ψ are continuously differentiable and $\varphi'(x) \neq 0$, $\psi'(x) \neq 0$ if $x \in I$.

Theorem 4. *Let $0 < \lambda < 1$ and $\mu \neq 0, 1$. If $\varphi, \psi \in \mathcal{CM}(I)$ solve (14) and φ and ψ are continuously differentiable and $\varphi'(x) > 0$, $\psi'(x) > 0$ for $x \in I$ then, with the notations*

$$J := \varphi(I), \quad f := \varphi' \circ \varphi^{-1}, \quad g := \psi' \circ \varphi^{-1}, \quad (15)$$

the continuous functions $f, g : J \rightarrow \mathbb{R}_+$ satisfy the functional equation

$$f(\lambda u + (1 - \lambda)v)(g(v) - g(u)) = \mu(f(u)g(v) - f(v)g(u)) \quad (16)$$

for all $u, v \in J$.

PROOF. Differentiate (14) with respect to x and y . (The assumptions of the theorem make this possible.) Then

$$\lambda = \mu \frac{\lambda \varphi'(x)}{\varphi'(A_\varphi(x, y; \lambda))} + (1 - \mu) \frac{\lambda \psi'(x)}{\psi'(A_\psi(x, y; \lambda))}$$

and

$$1 - \lambda = \mu \frac{(1 - \lambda) \varphi'(y)}{\varphi'(A_\varphi(x, y; \lambda))} + (1 - \mu) \frac{(1 - \lambda) \psi'(y)}{\psi'(A_\psi(x, y; \lambda))}$$

for all $x, y \in I$. Multiplying the first equation by $(1 - \lambda) \psi'(y)$ and the second one by $\lambda \psi'(x)$, then subtracting the equations, we obtain

$$\psi'(y) - \psi'(x) = \frac{\mu(\varphi'(x)\psi'(y) - \varphi'(y)\psi'(x))}{\varphi'(A_\varphi(x, y; \lambda))}$$

for all $x, y \in I$. Let $u = \varphi(x)$ and $v = \varphi(y)$ ($u, v \in J$) be arbitrary. Then, with the notations (15), we get that (16) holds. $\square$

In what follows, without loss of generality $\mu > 0$ ($\mu \neq 1$) can be assumed. Since in (14) either μ or $\mu - 1$ is positive, interchanging φ and ψ if necessary, we may examine only the case when $\mu > 0$ ($\mu \neq 1$). In this case, the roles of φ and ψ are not interchangeable.

Theorem 5. *Let $0 < \lambda < 1$ and $\mu > 0$, ($\mu \neq 1$). If the continuous functions $f, g : J \rightarrow \mathbb{R}_+$ satisfy the functional equation (16) then the following cases are possible:*

- (i) *If $\lambda \neq \frac{1}{2}$ then there exists $c \in \mathbb{R}_+$ such that $f(u) = c$ for all $u \in J$;*
- (ii) *If $\lambda = \frac{1}{2}$ and $\mu \notin \{\frac{1}{2}, 2\}$ then there exists $c \in \mathbb{R}_+$ such that $f(u) = c$ for all $u \in J$;*
- (iii) *If $\lambda = \frac{1}{2}$ and $\mu = \frac{1}{2}$ then there exist $a, b \in \mathbb{R}$ such that*

$$f(u) = au + b > 0 \quad \text{if } u \in J;$$

- (iv) *If $\lambda = \frac{1}{2}$ and $\mu = 2$ then there exist $a, b \in \mathbb{R}$ such that*

$$f(u) = \frac{1}{au + b} > 0 \quad \text{if } u \in J.$$

PROOF. With the strict mean $M(u, v) := \lambda u + (1 - \lambda)v$ the functional equation (16) has the form (3). Then, by Theorem 2, there exists $c > 0$ such that $f(u)^\mu g(u)^{1-\mu} = c$ for all $u \in J$. This implies that the continuous function $f : J \rightarrow \mathbb{R}_+$ satisfies the functional equation

$$f(\lambda u + (1 - \lambda)v) \left(f(u)^{\frac{\mu}{1-\mu}} - f(v)^{\frac{\mu}{1-\mu}} \right) = \mu \left(f(u)^{\frac{\mu}{1-\mu}+1} - f(v)^{\frac{\mu}{1-\mu}+1} \right)$$

for all $u, v \in J$. By Theorem 3, then either $f(u) = c$ ($u \in J$) for some $c \in \mathbb{R}_+$, or f is injective on J , and thus, for all $x = f(u)$, $y = f(v)$ ($x, y \in f(J) =: K \subset \mathbb{R}_+$) $x \neq y$

$$f(\lambda f^{-1}(x) + (1 - \lambda)f^{-1}(y)) = S_{\frac{\mu}{1-\mu}+1, \frac{\mu}{1-\mu}}(x, y) \quad (17)$$

holds. The following cases are possible:

(i) If $\lambda \neq \frac{1}{2}$ then f cannot be injective. Indeed, if it were so, then from the symmetry of the right-hand side of (17),

$$f(\lambda f^{-1}(x) + (1 - \lambda)f^{-1}(y)) = f(\lambda f^{-1}(y) + (1 - \lambda)f^{-1}(x))$$

would follow, i.e., $x = y$, which is a contradiction. Thus in this case f is constant on J .

(ii) If $\lambda = \frac{1}{2}$ and f is injective then from (17) we have

$$f\left(\frac{f^{-1}(x) + f^{-1}(y)}{2}\right) = S_{\frac{\mu}{1-\mu}+1, \frac{\mu}{1-\mu}}(x, y) \quad (18)$$

for all $x, y \in K \subset \mathbb{R}_+$. By Lemma 1, (18) holds if and only if either $\mu = \frac{1}{2}$ or $\mu = 2$. Thus if $\mu \notin \{\frac{1}{2}, 2\}$, f is constant on J .

(iii) If $\lambda = \frac{1}{2}$ and $\mu = \frac{1}{2}$ then we may suppose that f is injective. From (18) we have

$$f\left(\frac{f^{-1}(x) + f^{-1}(y)}{2}\right) = \frac{x + y}{2} \quad (x \neq y; x, y \in K),$$

which implies the existence of $a, b \in \mathbb{R}$ with $a \neq 0$ such that

$$f(u) = au + b > 0 \quad \text{if } u \in J.$$

If $a = 0$ then $f(u) = b > 0$ gives the constant solutions.

(iv) If $\lambda = \frac{1}{2}$ and $\mu = 2$ one can easily see that there exist $a, b \in \mathbb{R}_+$ for which

$$f(u) = \frac{1}{au+b} > 0 \quad \text{if } u \in J$$

because in this case the function $\frac{1}{f}$ satisfies the Jensen equation.

Thus, the proof of the theorem is complete. $\square$

Returning to the generalized Matkowski–Sutô problem, we need the following definitions.

Definition 1. Let $\varphi, \psi \in \mathcal{CM}(I)$. If there exist $a \neq 0$ and b such that

$$\psi(x) = a\varphi(x) + b \quad \text{if } x \in I$$

then we say that φ is equivalent to ψ on I and denote it by $\varphi(x) \sim \psi(x)$ if $x \in I$ or in short $\varphi \sim \psi$ on I . If $(\varphi, \psi) \in \mathcal{CM}(I)^2$, $(\Phi, \Psi) \in \mathcal{CM}(I)^2$ and $\varphi \sim \Phi$ and $\psi \sim \Psi$ on I then we say that the pair (φ, ψ) is equivalent to the pair (Φ, Ψ) on I and this fact is denoted by $(\varphi, \psi) \sim (\Phi, \Psi)$ on I .

The statement of the following lemma is well known, therefore its proof will be omitted.

Lemma 2. Let $0 < \lambda < 1$ and $\varphi, \psi \in \mathcal{CM}(I)$. Then $A_\varphi(x, y; \lambda) = A_\psi(x, y; \lambda)$ for all $x, y \in I$ if and only if $\varphi \sim \psi$ on I .

Definition 2. Let $I \subset \mathbb{R}$ be a nonvoid open interval. Define the one parameter family of functions $\chi_p : I \rightarrow \mathbb{R}$ ($p \in \mathbb{R}$) as follows

$$\chi_p(x) = \begin{cases} x & \text{if } p = 0 \\ e^{px} & \text{if } p \neq 0 \end{cases} \quad (x \in I).$$

We also define the following sets

$$\begin{aligned} P_+(I) &:= \{p \in \mathbb{R} \mid I + p \subset \mathbb{R}_+\} \\ P_-(I) &:= \{p \in \mathbb{R} \mid -I + p \subset \mathbb{R}_+\}. \end{aligned}$$

With the help of these notations, set

$$\begin{aligned} \gamma_p^+(x) &:= \sqrt{x+p} & \text{if } p \in P_+(I) & \quad (x \in I) \\ \gamma_p^-(x) &:= \sqrt{-x+p} & \text{if } p \in P_-(I) & \quad (x \in I). \end{aligned}$$

Finally, we introduce the following notations

$$\begin{aligned}\mathcal{S}(I) &:= \{(\chi_p, \chi_{-p}) \mid p \in \mathbb{R}\}, \\ \mathcal{T}(I) &:= \{(\chi_0, \chi_0)\} \cup \{(\gamma_p^+, \log \circ \gamma_p^+) \mid p \in P_+(I)\} \\ &\quad \cup \{(\gamma_p^-, \log \circ \gamma_p^-) \mid p \in P_-(I)\}.\end{aligned}$$

Theorem 6. *Let $0 < \lambda < 1$ and $\mu > 0$, ($\mu \neq 1$). If $\varphi, \psi \in \mathcal{CM}(I)$ solve the generalized Matkowski–Sutô problem (14) and φ, ψ are continuously differentiable with nonvanishing derivatives on I then the following cases are possible:*

- (i) *If $\lambda \neq \frac{1}{2}$ then $(\varphi, \psi) \sim (\chi_0, \chi_0)$ on I ;*
- (ii) *If $\lambda = \frac{1}{2}$ and $\mu \notin \{\frac{1}{2}, 2\}$ then $(\varphi, \psi) \sim (\chi_0, \chi_0)$ on I ;*
- (iii) *If $\lambda = \frac{1}{2}$ and $\mu = \frac{1}{2}$ then there exists $(s_1, s_2) \in \mathcal{S}(I)$ such that $(\varphi, \psi) \sim (s_1, s_2)$ on I ;*
- (iv) *If $\lambda = \frac{1}{2}$ and $\mu = 2$ then there exists $(t_1, t_2) \in \mathcal{T}(I)$ such that $(\varphi, \psi) \sim (t_1, t_2)$ on I .*

The pairs (φ, ψ) given in the cases (i), (ii), (iii), and (iv) are solutions of equation (14).

PROOF. By Lemma 2, it is enough solve the functional equation (14) up to the equivalence of the functions φ and ψ . Thus we may assume that $\varphi'(x) > 0$ and $\psi'(x) > 0$ if $x \in I$. Then, with the notations (15), Theorem 4 implies that (16) holds. Now using Theorem 5, we have that the cases (i)–(iv) are possible for $f := \varphi' \circ \varphi^{-1}$. Due to the definition of f , we obtain the differential equation for the function φ :

$$\varphi'(x) = f(\varphi(x)) \quad (x \in I). \quad (19)$$

In the cases (i) and (ii) (19) yields $\varphi(x) = cx + b$ ($x \in I$), that is, $\varphi \sim \chi_0$ on I . Furthermore, equation (14) results that $\psi \sim \chi_0$ on I .

In the case (iii), (19) reduces to the differential equation

$$\varphi'(x) = a\varphi(x) + b > 0 \quad (x \in I),$$

whence we get that there exists $p \in \mathbb{R}$ such that $\varphi \sim \chi_p$ on I . Now applying (14), we simply obtain that $\psi \sim \chi_{-p}$, i.e., $(\varphi, \psi) \sim (s_1, s_2)$ for some pair $(s_1, s_2) \in \mathcal{S}(I)$.

Finally, in the case (iv), (19) simplifies to the differential equation

$$\varphi'(x) = \frac{1}{a\varphi(x) + b} > 0 \quad (x \in I)$$

from which we deduce that either $\varphi \sim \chi_0$ on I , or there exists $p \in P_+(I)$ such that $\varphi \sim \gamma_p^+$ on I or there exists $p \in P_-(I)$ such that $\varphi \sim \gamma_p^-$ on I . Now applying (14), we get $(\varphi, \psi) \sim (t_1, t_2)$ for some pair $(t_1, t_2) \in \mathcal{T}(I)$. $\square$

The result of Theorem 6 can be restated in a different but equivalent form, too. For, we introduce the following notations.

If $I \subset \mathbb{R}$ is a nonvoid open interval then we define the following means:

If $p \in \mathbb{R}$ then set

$$T_p(x, y) := \begin{cases} \frac{x+y}{2} & \text{if } p = 0 \\ \frac{1}{p} \log \left(\frac{e^{px} + e^{py}}{2} \right) & \text{if } p \neq 0 \end{cases} \quad (x, y \in I). \quad (20)$$

For $p \in P_+(I)$ define

$$\Gamma_p^+(x, y) := \left(\frac{\sqrt{x+p} + \sqrt{y+p}}{2} \right)^2 - p \quad \text{and} \quad (21)$$

$$G_p^+(x, y) := \sqrt{(x+p)(y+p)} - p \quad (x, y \in I)$$

Finally, for $p \in P_-(I)$ define

$$\Gamma_p^-(x, y) := - \left(\frac{\sqrt{-x+p} + \sqrt{-y+p}}{2} \right)^2 + p \quad \text{and} \quad (22)$$

$$G_p^-(x, y) := - \sqrt{(-x+p)(-y+p)} + p \quad (x, y \in I).$$

One can easily check that the expressions given by (20), (21) and (22) are the quasi-arithmetic means on I generated by the functions χ_p in (20), γ_p^+ and $\log \gamma_p^+$ in (21) and γ_p^- and $\log \gamma_p^-$ in (22), respectively.

Theorem 7. *Let $0 < \lambda < 1$ and $\mu > 0$ ($\mu \neq 1$). Suppose that the quasi-arithmetic means $M, N : I^2 \rightarrow I$ weighted by λ satisfy the generalized Matkowski–Sutô equation*

$$\lambda x + (1 - \lambda)y = \mu M(x, y) + (1 - \mu)N(x, y) \quad (4)$$

for all $x, y \in I$. If the generator functions of M and N are continuously differentiable with nonvanishing derivatives on I then the following cases are possible:

(i) If $\lambda \neq \frac{1}{2}$ then

$$M(x, y) = N(x, y) = A(x, y; \lambda) \quad (x, y \in I);$$

(ii) If $\lambda = \frac{1}{2}$ and $\mu \notin \{\frac{1}{2}, 2\}$ then

$$M(x, y) = N(x, y) = A(x, y; 1/2) = A(x, y) \quad (x, y \in I);$$

(iii) If $\lambda = \frac{1}{2}$ and $\mu = \frac{1}{2}$ then

$$M(x, y) = T_p(x, y) \quad \text{and} \quad N(x, y) = T_{-p}(x, y) \quad (x, y \in I)$$

for some $p \in \mathbb{R}$;

(iv) If $\lambda = \frac{1}{2}$ and $\mu = 2$ then either

$$M(x, y) = N(x, y) = A(x, y) \quad (x, y \in I),$$

or

$$M(x, y) = \Gamma_p^+(x, y) \quad \text{and} \quad N(x, y) = G_p^+(x, y) \quad (x, y \in I)$$

for some $p \in P_+(I)$, or

$$M(x, y) = \Gamma_p^-(x, y) \quad \text{and} \quad N(x, y) = G_p^-(x, y) \quad (x, y \in I)$$

for some $p \in P_-(I)$.

Conversely, all the means M and N listed in the above cases are solutions of (4).

PROOF. Theorem 6 immediately implies the statements. $\square$

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(Received May 7, 2002)