

On weakly conformally symmetric Ricci-recurrent spaces

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Dedicated to Professor Lajos Tamássy on his 80th birthday

Abstract. The present paper is concerned with n -dimensional ($n > 3$) Riemannian spaces are weakly conformally symmetric. It is proved that such a space satisfying the second Bianchi identity can be endowed with a uniquely determined semi-symmetric metric connection with respect to which the conformal curvature tensor is weakly symmetric. Finally we study weakly conformally symmetric Ricci-recurrent spaces.

1. Introduction

The notion of weakly symmetric and weakly projective symmetric spaces was introduced by TAMÁSSY and BINH [1]. A non-flat Riemannian space V_n ($n > 2$) is called a weakly symmetric space if the curvature tensor R_{hijk} satisfies the condition:

$$R_{hijk,l} = a_l R_{hijk} + b_h R_{lij k} + d_i R_{hljk} + e_j R_{hilk} + f_k R_{hijl} \quad (1.1)$$

where a, b, d, e, f are 1-forms (non-zero simultaneously) and the comma ', ' denotes covariant differentiation with respect to the metric tensor of the space.

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The 1-forms a, b, d, e, f are called the associated 1-forms of the space, and an n -dimensional space of this kind is denoted by $(WS)_n$. Such a space have been studied by M. PRVANOVIĆ [2], T. Q. BINH [3], U. C. DE and S. BANDHYOPADHYAY [4] and others. In [4] DE and BANDHYAOPADHYAY proved that the associated 1-forms d and f are identical with b and e respectively. Hence $(WS)_n$ is characterized by the condition

$$R_{hijk,l} = a_l R_{hijk} + b_h R_{lij k} + b_i R_{hljk} + e_j R_{hil k} + e_k R_{hij l}. \quad (1.2)$$

In the same paper [4] DE and BANDHYOPADHYAY proved the existence of a $(WS)_n$ by considering a metric. In a subsequent paper [5] DE and BANDHYOPADHYAY introduce the notion of weakly conformally symmetric spaces.

An n -dimensional ($n > 3$) non-conformally flat Riemannian space is called weakly *conformally symmetric* if its conformal curvature tensor C_{hijk} defined by

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{hk}R_{ij} - g_{hj}R_{ik} + g_{ij}R_{hk} - g_{ik}R_{hj}) + \frac{R}{(n-1)(n-2)}(g_{hk}g_{ij} - g_{hj}g_{ik}) \quad (1.3)$$

satisfies the condition

$$C_{hijk,l} = a_l C_{hijk} + b_h C_{lij k} + d_i C_{hljk} + e_j C_{hil k} + f_k C_{hij l} \quad (1.4)$$

where a, b, d, e, f are 1-forms (non-zero simultaneously). Such a space is denoted by $(WCS)_n$. Since the conformal curvature tensor C_{hijk} satisfies the skew-symmetric property for the indices, h, i and j, k , like curvature tensor R_{hijk} also, the defining relation (1.4) can be expressed in the following form:

$$C_{hijk,l} = a_l C_{hijk} + b_h C_{lij k} + b_i C_{hljk} + e_j C_{hil k} + e_k C_{hij l}. \quad (1.5)$$

It is well known that the conformal curvature tensor satisfies the conditions

$$C_{ijk}^h + C_{jki}^h + C_{kij}^h = 0, \quad (1.6)$$

$$C_{rjk}^r = C_{irk}^r = C_{ijr}^r = 0, \quad (1.7)$$

$$C_{hijk} = -C_{hikj} = -C_{ihjk} = C_{jkhi}. \quad (1.8)$$

An n -dimensional Riemannian space is said to be Ricci-recurrent [6] if its Ricci tensor is non-zero and satisfies the conditions

$$R_{ij,k} = \lambda_k R_{ij}, \quad (1.9)$$

where λ_k is a non-zero 1-form.

If the 1-forms $a = b = e = o$ in (1.5), then the space is called conformally symmetric [7]. If the 1-forms are all equal, the space is called conformally quasi-recurrent space introduced by M. PRVANOVIĆ [8]. Conformally symmetric Ricci-recurrent spaces have been studied by W. ROTER [9].

2. Semi-symmetric metric connection

In general, the tensor C_{ijk}^h does not satisfy the second Bianchi identity

$$C_{ijk,l}^h + C_{ikl,j}^h + C_{ilj,k}^h = 0. \quad (2.1)$$

In this section we suppose that condition (2.1) holds in the investigated $(WCS)_n$.

Transvecting (1.5) with g^{sh} we obtain

$$C_{ijk,l}^s = a_l C_{ijk}^s + b^s C_{lij,k} + b_i C_{lj,k}^s + e_j C_{il,k}^s + e_k C_{ij,l}^s. \quad (2.2)$$

Using (1.6) and (1.8) from (2.1) and (2.2) it follows that

$$A_l C_{ijk}^h + A_j C_{ikl}^h + A_k C_{ilj}^h = 0, \quad (2.3)$$

where $A_i = a_i - 2e_i$.

Contracting h and l in (2.3) and using (1.7) we get

$$A_h C_{ijk}^h = 0. \quad (2.4)$$

Transvecting (2.3) with A^l and using (2.4) we obtain

$$A_l A^l C_{ijk}^h = 0. \quad (2.5)$$

Hence $A_l A^l = 0$, since by assumption $C_{ijk}^h \neq 0$. Thus we can state the following

Theorem 1. *The vector $A_i = a_i - 2e_i$ in a $(WCS)_n$ is a null vector.*

Now we shall prove

Theorem 2. *A weakly conformally symmetric space can be endowed with a uniquely determined semi-symmetric metric connection with respect to which the conformal curvature tensor is weakly conformally symmetric.*

PROOF. A connection of the form

$$\tilde{\Gamma}_{jk}^h = \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} + \delta_j^h S_k - g_{jk} S^h,$$

where S_i is a vector field, is a semi-symmetric metric connection. In fact, denoting the operator of covariant differentiation with respect to this connection by $\tilde{\nabla}$, we have

$$\tilde{\nabla}_k g_{ij} = \frac{\partial g_{ij}}{\partial x^k} - \tilde{\Gamma}_{kl}^r g_{rj} - \tilde{\Gamma}_{kj}^r g_{ir} = 0,$$

while the torsion tensor has the form

$$T_{jk}^i = \tilde{\Gamma}_{jk}^i - \tilde{\Gamma}_{kj}^i = S_k \delta_j^i - S_j \delta_k^i.$$

We shall denote the tensors determined with respect to $\tilde{\Gamma}$ by a tilde above. For example, $\tilde{R}_{ijk}^h$ is the curvature tensor of this connection, $\tilde{R}_{ij}$ is the Ricci tensor, $\tilde{R}$ is the scalar curvature, while

$$\begin{aligned} \tilde{C}_{ijk}^h &= \tilde{R}_{ijk}^h - \frac{1}{n-2} (g_{ij} \tilde{R}_k^h - g_{ik} \tilde{R}_j^h + \delta_k^h \tilde{R}_{ij} - \delta_j^h \tilde{R}_{ik}) \\ &\quad + \frac{\tilde{R}}{(n-1)(n-2)} (\delta_k^h g_{ij} - \delta_j^h g_{ik}). \end{aligned}$$

It is known that [10]

$$\tilde{C}_{ijk}^h = C_{ijk}^h. \quad (2.6)$$

Applying the operator $\tilde{\nabla}$ to (2.6) and using (2.2), we get

$$\begin{aligned} \tilde{\nabla} \tilde{C}_{ijk}^h &= a_s C_{ijk}^h + b^h C_{sijk} + b_i C_{sjk}^h + e_j C_{isk}^h + e_k C_{ijs}^h - S^h C_{sijk} \\ &\quad - S_i C_{sjk}^h - S_j C_{isk}^h - S_k C_{ijs}^h + \delta_s^h S_r C_{ijk}^r + g_{is} S^r C_{rjk}^h \\ &\quad + g_{js} S^r C_{irk}^h + g_{ks} S^r C_{ijr}^h. \end{aligned} \quad (2.7)$$

Therefore, if $S_i = A_i$, i.e., if

$$\tilde{\Gamma}_{jk}^h = \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} + \delta_j^h A_k - g_{jk} A^h,$$

and if we take into account (2.4) and (2.6), we find,

$$\begin{aligned} \tilde{\nabla}_s C_{ijk}^h &= a_s C_{ijk}^h + (b^h - A^h) C_{sijk} + (b_i - A_i) C_{sjk}^h \\ &\quad + (e_j - A_j) C_{isk}^h + (e_k - A_k) C_{ijs}^h \end{aligned} \quad (2.8)$$

which implies that the conformal curvature tensor is weakly conformally symmetric with respect to the connection $\tilde{\nabla}$. This completes the proof of the theorem. $\square$

If, in particular, $A_i = b_i = e_i$, then (2.8) reduces to $\tilde{\nabla}_s C_{ijk}^h = a_s C_{ijk}^h$. Hence we can state the following

Corollary. *A weakly conformally symmetric space can be endowed with a uniquely determined semi-symmetric metric connection with respect to which the conformal curvature tensor is recurrent if the associated vector S_i of the semi-symmetric connection is equal to the associated vectors b_i and e_i of a weakly conformally symmetric space.*

3. Weakly conformally symmetric Ricci-recurrent space

A Ricci-recurrent space is defined by (1.6). Transvecting (1.6) with g^{ij} , we obtain

$$R_{,k} = \lambda_k R. \quad (3.1)$$

Hence if $R \neq 0$, the recurrence vector λ_k is gradient. W. ROTER [9] proved that in a Ricci-recurrent space the following relations

$$R_{ri} R_{jlm}^r + R_{rj} R_{ilm}^r = 0 \quad (3.2)$$

and

$$R_{ri} R_j^r = \frac{1}{2} R R_{ij} \quad (3.3)$$

hold.

In view of the known result $R_{j,s}^s = \frac{1}{2}R_{,j}$ (1.6) implies that

$$\lambda_s R_j^s = \frac{1}{2} R \lambda_j. \quad (3.4)$$

Transvecting (1.5) with g^{sh} we obtain

$$C_{ijk,l}^s = a_l C_{ijk}^s + b^s C_{lijk} + b_i C_{ljk}^s + e_j C_{ilk}^s + e_k C_{ijl}^s.$$

Since $C_{sjk}^s = C_{jsk}^s = C_{jks}^s = 0$, the above equation reduces to

$$C_{ijk,s}^s = T_s C_{ijk}^s, \quad (3.5)$$

where $T_s = a_s + b_s$.

It is known that in a Riemannian space [11, p. 91]

$$C_{ijk,s}^s = \frac{n-3}{n-2} \left[\left(R_{ij,k} - \frac{1}{2(n-1)} R_{,k} g_{ij} \right) - \left(R_{ik,j} - \frac{1}{2(n-1)} R_{,j} g_{ik} \right) \right]$$

which in view of (1.6) and (3.4) yields,

$$\begin{aligned} T_s C_{ijk}^s &= \frac{n-3}{n-2} \left[\left(R_{ij} - \frac{1}{2(n-1)} R g_{ij} \right) \lambda_k \right. \\ &\quad \left. - \left(R_{ik} - \frac{1}{2(n-1)} R g_{ik} \right) \lambda_j \right]. \end{aligned} \quad (3.6)$$

Since $T_s T^i C_{ijk}^s = 0$, equation (3.6) implies

$$\left(T_s R_j^s - \frac{1}{2(n-1)} R T_j \right) \lambda_k = \left(T_s R_k^s - \frac{1}{2(n-1)} R T_k \right) \lambda_j. \quad (3.7)$$

Transvecting (3.7) with R_p^k , using (3.3) and (3.4), and then comparing it with (3.7) we get

$$T_s R_j^s = \frac{1}{2} R T_j. \quad (3.8)$$

Transvecting (1.5) with T^h and by virtue of (3.6) and (3.8) we get

$$\begin{aligned} &\frac{n-3}{n-2} \left[\left(R_{ij} - \frac{1}{2(n-1)} R g_{ij} \right) \lambda_k - \left(R_{ik} - \frac{2}{2(n-1)} R g_{ik} \right) \lambda_j \right] \\ &= T_s R_{ijk}^s - \frac{1}{n-2} (T_k R_{ij} - T_j R_{ik}) \end{aligned} \quad (3.9)$$

$$+ \frac{3-n}{2(n-1)(n-2)} R(T_k g_{ij} - T_j R_{ik}).$$

Now

$$\begin{aligned} T_s R_{ijk}^s R_p^i &= T^s R_p^i R_{sijk} = -T^s R_p^i R_{isjk} \\ &= -T^s R_{ip} R_{sjk}^i = T^s R_{is} R_{pjk}^i, \quad \text{by (3.2)} \\ &= \frac{1}{2} R T_i T_{pjk}^i \quad \text{by (3.8)}. \end{aligned}$$

Transvecting (3.9) with R_p^i and using the above relation and (3.3) we obtain

$$\frac{n-3}{n-1} (\lambda_k R_{jp} - \lambda_j R_{kp}) = T_s R_{pjk}^s - \frac{2}{n-1} (T_k R_{jp} - T_j R_{kp})$$

which, by contraction with g^{pj} yields

$$\frac{n-3}{n-1} (R\lambda_k - \lambda_j R_k^j) = T_s R_k^s - \frac{2}{n-1} (T_k R - T_j R_k^j).$$

Now using (3.4) and (3.8) we obtain $T_k = \lambda_k$.

Thus we can state the following

Theorem 3. *In a weakly conformally symmetric Ricci-recurrent space with $R \neq 0$, the relation $\lambda_k = a_k + b_k$ holds.*

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