

Perturbations of nonlinear evolution equations

By W. M. BIAN (Southampton)

Abstract. Existence results are given for the evolution inclusions $x'(t) + A(t, x(t)) + G(t, x(t)) \ni f(t)$ with $A(t, \cdot)$ a monotone mapping and G a set-valued bounded or Lipschitz mapping.

1. Introduction

Consider the existence of solutions to the evolution inclusion

$$x'(t) + A(t, x(t)) + G(t, x(t)) \ni f(t) \text{ a.e. on } [0, T], \quad x(0) = x_0$$

in a evolution triple (V, H, V^*) with $A(t, \cdot)$ a monotone mapping and G a set-valued mapping.

As a perturbation to the classical equation $x'(t) + A(x(t)) = f(t)$, this problem is very important not only in evolution equation theory but also in other subjects such as distributed parameter control systems (see [1], [2], [11]). So, it has been recently studied in many publications under different conditions (see [3], [6], [8], [9], [12] and the references therein). In [3], the coerciveness assumptions made to A involves the norm of H and G is assumed to satisfy a convergence condition; In [8] and [9], $v \mapsto G(t, v)$ is supposed to be an upper semicontinuous mapping with closed convex values and satisfy a growth condition. In [6], G can be an nonconvex-valued mapping but it is supposed to be integrably bounded. We notice

Mathematics Subject Classification: 34G25, 34G20, 47H05.

Key words and phrases: nonlinear evolution inclusions, monotone mappings, perturbations.

although a wrong imbedding result is used in [6], the main conclusion is not effected due to the reason stated in [8].

In this paper, we will give two new existence results for the above problem. In one of our result, we just suppose that the mapping $v \mapsto G(t, v)$ is upper semicontinuous and bounded (maps bounded sets into bounded sets), do not impose growth condition, and consider the local existence. In another result, $v \mapsto G(t, v)$ is supposed to be Lipschitz with the constant depending on t and the values of G can be nonconvex. A continuity theorem is also presented, which is a modification to a similar one in [8].

2. Preliminaries

In this paper, we always suppose (V, H, V^*) is an evolution triple, that is, H is Hilbert space, V is a separable reflexive Banach space with dual V^* and $V \hookrightarrow H \hookrightarrow V^*$ densely and continuously. The inner product of H as well as the duality pairing between V and V^* are denoted by $(\cdot, \cdot)$. We also suppose that $\infty > p \geq 2$ is a real number and $q = p/(p-1)$. $((\cdot, \cdot))$ stands for the duality pairing between $L^p(0, T; V)$ and $L^q(0, T; V^*)$. The norm in any Banach space X involved is denoted by $\|\cdot\|_X$. The space X endowed with weak topology is denoted by X_w , the weak convergence (in X) is denoted by " $x_n \rightharpoonup x$ ", and the functional space $L^r(0, T; X)$ with $r > 0$ will be abbreviated to $L^r(X)$. For a sequence of subsets $D_n \subset X$, we denote by

$$w\text{-}\limsup_{n \rightarrow \infty} D_n = \{x \in X : \text{there exist } n_k \text{ and } x_{n_k} \in D_{n_k} \text{ with } x_{n_k} \rightharpoonup x\}.$$

For a set-valued mapping $G : [0, T] \rightarrow X$, we denote by

$$S_G^1 = \{x \in L^1(X) : x(t) \in G(t) \text{ a.e.}\}.$$

A known result is that $S_G^1 \neq \emptyset$ if $\inf\{\|u\|_X : u \in G(t) \text{ a.e.}\} \in L^1(X)$.

Let $W(0, T) = \{x \in L^p(V) : x' \in L^q(V^*)\}$. It is known that $W(0, T)$ is a reflexive Banach space endowed with the norm $\|x\|_W := \|x\|_{L^p(V)} + \|x'\|_{L^q(V^*)}$, $W(0, T) \hookrightarrow C(0, T; H)$ continuously and, if the imbedding of V into H is compact, then $W(0, T) \hookrightarrow L^p(H)$ compactly.

We also recall that a set-valued mapping F between Hausdorff spaces X and Y is said to be upper semicontinuous (u.s.c.) if $F^{-1}(D) := \{x \in X : F(x) \cap D \neq \emptyset\}$ is closed for each closed subset $D \subset Y$; F is said to be hemicontinuous if $t \mapsto F(x + ty)$ is u.s.c. If X is a reflexive Banach space, $Y = X^*$ and $(y_1 - y_2, x_1 - x_2) \geq 0$ for all $x_i \in X, y_i \in F(x_i), i = 1, 2$, then F is called monotone on X .

Now, we consider evolution equation

$$x'(t) + A(t, x(t)) = f(t) \text{ a.e. on } [0, T], \quad x(0) = x_0 \in H \quad (2.1)$$

under the following assumptions.

(H1) $A : [0, T] \times V \rightarrow V^*$ is an operator with $t \mapsto A(t, v)$ measurable, $v \mapsto A(t, v)$ hemicontinuous and monotone.

(H2) There exist $a_1 \geq 0, a_2 \in L^q(0, T)$ such that

$$\|A(t, v)\|_{V^*} \leq a_1 \|v\|_V^{p-1} + a_2(t), \quad \text{for all } v \in V, t \in [0, T].$$

(H3) There exist $a_3 > 0, a_4 \in L^1(0, T)$ such that

$$(A(t, v), v) \geq a_3 \|v\|_V^p - a_4(t), \quad \text{for all } v \in V, t \in [0, T].$$

(H4) $V \hookrightarrow H$ compactly.

It is well known (see [7] or [12]) that if (H1)–(H3) are satisfied, then, for each $x_0 \in H$ and each $f \in L^q(V^*)$, equation (2.1) has a unique solution in $W(0, T)$, which will be always denoted in the following by x_f , and, if D is a bounded subset of $L^q(V^*)$, then the solution set $\{x_f : f \in D\}$ is bounded in $W(0, T)$. In fact, there exists $c > 0$ such that $\|x_f\|_{W(0, T)} \leq c + c\|f\|_{L^q(V^*)}$. (This is also true for some implicit problems, see [4].) Moreover, the solution mapping $f \mapsto x_f$ has a property as stated below.

Proposition 2.1. *Suppose (H1)–(H4) are satisfied. Then the solution mapping $f \mapsto x_f$ of equation (2.1) is continuous from $L^q(H)_w$ to $C(0, T; H)$, monotone on $L^q(V^*)$ and*

$$\|x_f(t) - x_g(t)\|_H \leq \int_0^t \|f(s) - g(s)\|_H ds \quad \text{for all } f, g \in L^q(H). \quad (2.2)$$

PROOF. Let $f_n \rightharpoonup f$ in $L^q(H)$. Then $\{f_n\}$ is bounded in $L^q(V^*)$. From the remarks we made above, we know that $\{x_{f_n}\}$ is bounded in $W(0, T)$.

So, by passing to a subsequence, we may assume that $x_{f_n} \rightharpoonup y$ in $W(0, T)$. Since $W(0, T) \hookrightarrow C(0, T; H)$ continuously, $\{x_{f_n}\}$ is bounded in $C(0, T; H)$. Since $x_{f_n}(0) = x_f(0) = x_0$ and

$$x'_{f_n}(t) + A(t, x_{f_n}(t)) = f_n(t), \quad x'_f(t) + A(t, x_f(t)) = f(t) \quad \text{a.e.,}$$

$$(x'_{f_n}(t) - x'_f(t), x_{f_n}(t) - x_f(t)) = \frac{1}{2} \frac{d}{dt} \|x_{f_n}(t) - x_f(t)\|_H^2,$$

by the monotonicity of $A(t, \cdot)$, we have

$$\frac{1}{2} \frac{d}{dt} \|x_{f_n}(t) - x_f(t)\|_H^2 \leq (f_n(s) - f(s), x_{f_n}(s) - x_f(s)).$$

Therefore

$$\frac{1}{2} \|x_{f_n}(t) - x_f(t)\|_H^2 \leq \int_0^t (f_n(s) - f(s), x_{f_n}(s) - x_f(s)) ds \quad (2.3)$$

$$\begin{aligned} &= \int_0^t (f_n(s) - f(s), x_{f_n}(s) - y(s)) ds \\ &\quad + \int_0^t (f_n(s) - f(s), y(s) - x_f(s)) ds. \end{aligned} \quad (2.4)$$

Since $W(0, T) \hookrightarrow L^p(H)$ compactly, we may suppose that $x_{f_n} \rightarrow y$ strongly in $L^p(H)$. So from the boundedness of $\{f_n\}$, it follows that

$$\int_0^t (f_n(s) - f(s), x_{f_n}(s) - y(s)) ds \leq \|f_n - f\|_{L^q(H)} \|x_{f_n} - y\|_{L^p(H)} \rightarrow 0.$$

By letting $\chi(s) = 1$ for $s \leq t$ and $\chi(s) = 0$ for $s > t$, we see

$$\begin{aligned} &\int_0^t (f_n(s) - f(s), y(s) - x_f(s)) ds \\ &= \int_0^T (f_n(s) - f(s), \chi(s)(y(s) - x_f(s))) ds \rightarrow 0. \end{aligned}$$

So, from (2.3) and (2.4), it follows that $\|x_{f_n}(t) - x_f(t)\|_H \rightarrow 0$ for each t . Together with the boundedness of $\{x_{f_n}\}$ in $C(0, T; H)$, we see that $x_{f_n} \rightarrow x_f$ in $L^p(H)$ and therefore, by (2.3) and Hölder's Inequality, we see

$$\|x_{f_n}(t) - x_f(t)\|_H^2 \leq 2\|f_n - f\|_{L^q(H)} \|x_{f_n} - x_f\|_{L^p(H)} \rightarrow 0.$$

That is, $x_{f_n}(t) \rightarrow x_f(t)$ in H uniformly. This proves the continuity of $f \mapsto x_f$ from $L^q(H)_w$ to $C(0, T; H)$.

Using the same method as used to obtain (2.3), we can prove, for all $f, g \in L^q(V^*)$, that

$$\frac{1}{2} \|x_f(t) - x_g(t)\|_H^2 \leq \int_0^t (f(s) - g(s), x_f(s) - x_g(s)) ds, \quad t \in [0, T]. \quad (2.5)$$

Let $t = T$, we see that $((f - g, x_f - x_g)) \geq 0$ which implies the monotonicity of $f \mapsto x_f$. If, in (2.5), let $f, g \in L^q(H)$, then we obtain

$$\frac{1}{2} \|x_f(t) - x_g(t)\|_H^2 \leq \int_0^t \|f(s) - g(s)\|_H \|x_f(s) - x_g(s)\|_H ds.$$

Applying the extended Gronwall's inequality (see [5] or [13]), we have

$$\|x_f(t) - x_g(t)\|_H \leq \int_0^t \|f(s) - g(s)\|_H ds.$$

This proves (2.2) and completes the proof. $\square$

Remark 2.2. The continuity of $f \mapsto x_f$ from $L^q(H)_w$ to $C(0, T; H)$ was also claimed in Proposition 1 of [8] where a_2 is a constant and $a_4 \equiv 0$. Moreover, our method is different.

3. Existence results

In this section, under (H1)–(H4), we suppose $G(t, \cdot)$ is either a bounded or a Lipschitz mapping on H , and consider the existence of solutions of the inclusion

$$x'(t) + A(t, x(t)) + G(t, x(t)) \ni f(t) \text{ a.e. on } [0, T], \quad x(0) = x_0 \in H. \quad (3.1)$$

Theorem 3.1. *Under assumptions (H1)–(H4), suppose $b \in L^q(0, T)$ is a given function. Let $G : [0, T] \times H \rightarrow 2^H$ be a set-valued mapping with closed convex values, $t \mapsto G(t, v)$ be measurable and $v \mapsto G(t, v)$ be u.s.c. from H to H_w . If for any bounded subset $D \subset H$, there exists $M > 0$ such that*

$$\sup\{\|G(t, v)\|_H : v \in D\} \leq M + b(t) \quad \text{a.e.,}$$

then problem (3.1) admits solutions on $[0, T_0]$ for some $T_0 \in (0, T]$.

PROOF. Let

$$d = \left(\|x_0\|_H^2 + 2\|a_4\|_{L^1(0,T)} + \frac{2}{q(pa_3)^{q/p}} \|f\|_{L^q(V^*)}^q \right)^{1/2} + \int_0^T b(t) dt,$$

$$D = \{u \in H : \|u\|_H \leq d + k\} \quad \text{with } k > 0 \text{ a given number.}$$

By our assumptions on G , there exists $M > 0$ such that

$$\sup\{\|u\|_H : u \in G(t, v), v \in D\} \leq M + b(t) \quad \text{a.e. on } [0, T]. \quad (3.2)$$

We choose $T_0 \in (0, T]$ such that $T_0 M \leq k$ (that is, $T_0 = \min\{T, k/M\}$) and denote by

$$D_1 = \{g \in L^q(H) : \|g(t)\|_H \leq M + b(t) \text{ a.e. on } [0, T_0]\},$$

$$F(g) = S_{G(\cdot, x_{f-g}(\cdot))}^1 \quad \text{for } g \in D_1.$$

Then D_1 is a bounded, closed and convex subset of $L^q(0, T_0; H)$, $F(g)$ is a nonempty, closed, bounded and convex subset for each $g \in D_1$.

Take $g \in D_1$ and write $x = x_{f-g}$ for convenience. Then

$$(x'(t), x(t)) + (A(t, x(t)), x(t)) = (f(t) - g(t), x(t)) \quad \text{a.e.}$$

From (H3), the fact that $(x'(t), x(t)) = \frac{1}{2} \frac{d}{dt} \|x(t)\|_H^2$ and Young's inequality, it follows that

$$\begin{aligned} \|x(t)\|_H^2 + 2a_3 \int_0^t \|x(s)\|_V^p ds &\leq \|x_0\|_H^2 + 2 \int_0^t a_4(s) ds \\ &+ 2 \int_0^t \|f(s)\|_{V^*} \|x(s)\|_V + 2 \int_0^t \|g(s)\|_H \|x(s)\|_H ds \leq \|x_0\|_H^2 \\ &+ 2\|a_4\|_{L^1(0,T)} + 2a_3 \int_0^t \|x(s)\|_V^p ds + \frac{2}{q(pa_3)^{q/p}} \int_0^t \|f(s)\|_{V^*}^q ds \\ &+ 2 \int_0^t \|g(s)\|_H \|x(s)\|_H ds. \end{aligned}$$

By the extended Gronwall's Inequality ([5] or [13]), we have

$$\|x(t)\|_H \leq \left(\|x_0\|_H^2 + 2\|a_4\|_{L^1(0,T)} + \frac{2}{q(pa_3)^{q/p}} \|f\|_{L^q(V^*)}^q \right)^{1/2}$$

$$+ \int_0^t \|g(s)\|_H ds \leq d + T_0 M \leq d + k \quad \text{a.e. on } [0, T_0].$$

So $x(t) = x_{f-g}(t) \in D$ for each $t \in [0, T_0]$ and, therefore, $\|z(t)\|_H \leq M + b(t)$ for each $z \in F(g)$ and each $t \in [0, T_0]$ because of (3.2). This means that F maps D_1 into itself as a set-valued mapping.

Let $(g_n, z_n) \in \text{Graph}(F)$ and $g_n \rightharpoonup g$, $z_n \rightharpoonup z$ in $L^q(0, T_0; H)$. By Proposition 2.1, $x_{f-g_n} \rightarrow x_{f-g}$ in $C(0, T_0; H)$ and, therefore, $x_{f-g_n}(t) \rightarrow x_{f-g}(t)$ in H for each $t \in [0, T_0]$. Since $G(t, \cdot)$ is u.s.c., we see

$$w\text{-}\limsup_{n \rightarrow \infty} G(t, x_{f-g_n}(t)) \subset G(t, x_{f-g}(t)) \quad \text{a.e.}$$

Invoking Theorem 4.2 of [10], we have

$$z \in w\text{-}\limsup_{n \rightarrow \infty} F(g_n) \subset S^1_{w\text{-}\limsup_{n \rightarrow \infty} G(\cdot, x_{f-g_n}(\cdot))} \subset S^1_{G(\cdot, x_{f-g}(\cdot))} = F(g).$$

So $(g, z) \in \text{Graph } F$, that is, F is closed under the weak topology. Since D_1 is weakly compact, we see that F is weakly u.s.c. under the weak topology. From Kakutani's fixed point theorem, it follows that F has fixed point, say g . By the meaning of the notion x_f we see that x_{f-g} is a solution of (3.1) on $[0, T_0]$. $\square$

Remark 3.2. Suppose x_1 is a solution of (3.1) on $[0, T_0]$. Then, by the same method as used above, we can prove that there exist $T_1 \in (T_0, T]$ and $x_2 \in W(T_0, T_1)$ such that

$$x_2(T_0) = x_1(T_0), \quad x_2'(t) + A(t, x_2(t)) + G(t, x_2(t)) \ni f(t) \quad \text{a.e. on } [T_0, T_1].$$

This implies that the interval on which (3.1) has solutions can be extended. But, without further assumptions, we are not sure whether this interval can be extended to $[0, T]$.

Now, we consider the case when G is Lipschitz with nonconvex values.

Theorem 3.3. *Under assumptions (H1)–(H4), let $G : [0, T] \times H \rightarrow 2^H$ be a set-valued mapping with closed and bounded values, $\sup\{\|u\|_H : u \in G(t, 0)\} \in L^q(0, T)$ and $t \mapsto G(t, v)$ be measurable. Suppose there exists $k \in L^q(0, T)$ such that*

$$\mathcal{H}(G(t, v_1), G(t, v_2)) \leq k(t)\|v_1 - v_2\|_H, \quad \text{for all } t \in [0, T], \quad v_1, v_2 \in H.$$

Here, $\mathcal{H}(\cdot, \cdot)$ means the Hausdorff distance on H . Then problem (3.1) has solutions. If, in addition, G is single-valued, then the solution is unique.

PROOF. Let $f \mapsto x_f$ be the same operator as in Proposition 2.1 and let

$$F(g) = S_{G(\cdot, x_{f-g}(\cdot))}^1 \quad \text{for } g \in L^q(H).$$

Then $F(g) \neq \emptyset$ for every $g \in L^q(H)$ and $F(g) \subset L^q(H)$ because of our assumptions on G . It is easy to see that $F(g)$ is closed and bounded.

Take $g_1, g_2 \in L^q(H)$ and let $\varepsilon > 0, z_1 \in F(g_1)$ be given. Since G is Lipschitz, there exists $z_2 \in F(g_2)$ such that

$$\|z_1(t) - z_2(t)\|_H \leq k(t)\|x_{f-g_1}(t) - x_{f-g_2}(t)\|_H + \varepsilon, \quad \text{a.e.}$$

Let $l > 0$ be a real number such that $2T^{1/p}(2lq)^{-q} < 1$. For each $z \in L^q(H)$, let

$$\|z\|_l = \left(\int_0^T \exp(-2lqr(t)) \|z(t)\|_H^q dt \right)^{1/q} \quad \text{with } r(t) = \int_0^t k^q(s) ds.$$

Clearly, $\|\cdot\|_l$ is a norm on $L^q(H)$ and equivalent to the usual one. By Proposition 2.1, Hölder's Inequality and using the integration by parts, we obtain

$$\begin{aligned} \|z_1 - z_2\|_l^q &= \int_0^T \exp(-2lqr(t)) \|z_1(t) - z_2(t)\|_H^q dt \\ &\leq 2^q \int_0^T \exp(-2lqr(t)) \left(k(t) \int_0^t \|g_1(s) - g_2(s)\|_H ds \right)^q dt \\ &\quad + \varepsilon 2^q \int_0^T \exp(-2lqr(t)) dt \\ &\leq 2^q T^{q/p} \int_0^T \exp(-2lqr(t)) k^q(t) \int_0^t \|g_1(s) - g_2(s)\|_H^q ds dt + 2^q \varepsilon T \\ &= -2^q \frac{T^{q/p}}{2lq} \exp(-2lqr(t)) \int_0^t \|g_1(s) - g_2(s)\|_H^q ds \Big|_0^T \\ &\quad + 2^q \frac{T^{q/p}}{2lq} \int_0^T \exp(-2lqr(s)) \|g_1(s) - g_2(s)\|_H^q ds + 2^q \varepsilon T \\ &\leq 2^q \frac{T^{q/p}}{2lq} \|g_1 - g_2\|_l^q + 2^q \varepsilon T. \end{aligned}$$

We denote by $\mathcal{H}_l(\cdot, \cdot)$ the Hausdorff distance in $L^q(H)$ endowed with the new norm $\|\cdot\|_l$. Since g_1, g_2 are arbitrary, we see

$$(\mathcal{H}_l(F(g_1), F(g_2)))^q \leq 2^q \frac{T^{q/p}}{2lq} \|g_1 - g_2\|_l^q + 2^q \varepsilon T.$$

By letting $\varepsilon \rightarrow 0$, we obtain

$$\mathcal{H}_l(F(g_1), F(g_2)) \leq 2T^{1/p}(2lq)^{-q} \|g_1 - g_2\|_l.$$

So F is a contraction on $L^q(H)$, and therefore F has a fixed point g . Obviously, $x = r(g)$ is a solution of (3.1). If, in addition, G is single-valued, then the solution is unique due to the uniqueness of fixed point of F as a single-valued mapping. $\square$

ACKNOWLEDGEMENTS. The author thanks the referees for valuable suggestion which improved the presentation of this paper.

References

- [1] N. U. AHMED, Optimal control of infinite dimensional systems governed by functional differential inclusions, *Discussiones Mathematicae, Differential Inclusions* **15** (1995), 75–94.
- [2] N. U. AHMED, Existence of optimal control for a class of systems governed by differential inclusions on a Banach space, *J. Optimization Theory Appl.* **50** (1986), 213–237.
- [3] N. U. AHMED, Compactness in certain abstract function spaces with application to differential inclusions, *Discussiones Mathematicae, Differential Inclusions* **15** (1995), 21–28.
- [4] W. M. BIAN, Solutions of implicit evolution inclusions with pseudo-monotone mappings, *Topological Methods Nonl. Anal.* **15** (2000), 101–113.
- [5] C. M. DAFERMOS, The second law of thermodynamics and stability, *Arch. Rational Mech. Anal.* **70** (1979), 167–179.
- [6] D. KRAVARITIS and G. PANTELIDIS, On evolution inclusions with nonconvex valued orientor fields, *Nonlinear Analysis, TMA* **26** (1996), 1–8.
- [7] J. L. LIONS, Quelques méthodes de résolution des problèmes aux limites non linéaires, *Dunod, Paris*, 1969.
- [8] S. MIGÓRSKI, On an existence result for nonlinear evolution inclusions, *Proc. Edin. Math. Soc.* **39** (1996), 133–141.
- [9] N. S. PAPAGEORGIOU, Continuous dependence results for evolution inclusions, *Proc. Edin. Math. Soc.* **35** (1992), 139–158.

- [10] N. S. PAPAGEORGIOU, Convergence theorems for Banach valued integrable multi-functions, *Intern. J. Math. Math. Sci.* **10** (1987), 433–442.
- [11] N. S. PAPAGEORGIOU, On Nonlinear nonconvex evolution inclusions, *Discussiones Mathematicar, Differential Inclusions* **15** (1995), 29–42.
- [12] R. E. SHOWALTER, Monotone Operator in Banach space and nonlinear partial differential equations, *Mathematical Surveys and Monographs* **49**, Amer. Math. Soc., Providence, R.I. (1997).
- [13] J. R. L. WEBB, An extension of Gronwall’s inequality, Proceeding of Conference on Nonlinear Partial Differential Equations, *Kiev*, 1997.

W. M. BIAN
DEPARTMENT OF ELECTRONICS & COMPUTER SCIENCE
UNIVERSITY OF SOUTHAMPTON
SOUTHAMPTON, SO17 1BJ
UNITED KINGDOM
E-mail: wb@ecs.soton.ac.uk

(Received July 7, 2001; revised August 13, 2002)