

**A class of contact Riemannian manifolds
whose associated CR-structures are integrable**

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Abstract. We study the geometry of contact Riemannian manifolds whose associated CR-structures are integrable.

1. Introduction

There are two typical examples of contact manifolds; one is formed by the principal circle bundles over symplectic manifolds of integral class (including the odd-dimensional spheres) and the other is given by the unit tangent sphere bundles of Riemannian manifolds. These spaces have the standard Riemannian structures and their associated CR-structures. A contact structure η is a global differentiable 1-form on a smooth manifold M^{2n+1} such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M . It is well-known that there exists an associated Riemannian metric structure g and $(1, 1)$ -type tensor ϕ , where (η, g) and (η, ϕ) are canonically related. We call the pair (η, g) a contact Riemannian structure and $M = (M; \eta, g)$ a contact Riemannian manifold. SASAKI and HATAKEYAMA [10] defined the normality of the contact Riemannian structure (see Section 2). A normal contact Riemannian manifold is said to be a Sasakian manifold. One the other

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hand, the associated CR-structure of a given contact Riemannian manifold $M = (M; \eta, g, \phi)$ is given by the holomorphic subbundle

$$\mathcal{H} = \{X - i\bar{\phi}X : X \in D\}$$

of the complexification $TM^{\mathbb{C}}$ of the tangent bundle TM , where D is the subbundle of TM defined by the kernel of η and $\bar{\phi} = \phi|_D$, the restriction of ϕ to D . Then we see that each fiber $\mathcal{H}_x$ ($x \in M$) is of complex dimension n and $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$. Furthermore, we have $\mathbb{C}D = \mathcal{H} \oplus \bar{\mathcal{H}}$. We say that *the associated CR-structure is integrable* if $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$. For $\mathcal{H}$ we define the Levi form by

$$L : D \times D \rightarrow \mathcal{F}(M), \quad L(X, Y) = -d\eta(X, \phi Y)$$

where $\mathcal{F}(M)$ denotes the algebra of differential functions on M . Then we see that the Levi form is hermitian and positive definite, that is, the CR-structure is a *strongly pseudo-convex, pseudo-hermitian CR structure*. In [14] S. TANNO proved that for a given contact Riemannian manifold M the associated CR-structure is strongly pseudo-convex integrable if and only if M satisfies the integrability condition $Q_1 = 0$ (see Section 2). Here, we note that the normality of a contact Riemannian structure implies the integrability of the associated CR-structure, but the converse does not always hold. The associated CR-structures of 3-dimensional contact Riemannian manifolds are always integrable (see [14]).

In this paper, we concentrate our attention on contact Riemannian manifolds whose associated CR-structures are integrable. We call such manifolds briefly, *contact Riemannian manifolds of class $\mathfrak{Q}_1$* . We introduce a class $\mathfrak{Q}$ of contact Riemannian manifolds whose associated CR-structures are integrable ($Q_1 = 0$) and satisfy the additional condition $Q_2 = 0$ (see Section 4). Here, we remark that these classes $\mathfrak{Q}_1$ and $\mathfrak{Q}$ are invariant under a D -homothetic deformation (see Sections 2, 4). Further, we can see that the class $\mathfrak{Q}$ contains the unit tangent sphere bundles of real space forms and the contact (k, μ) -spaces which appeared in [3], [6]. In Section 2, we prepare some fundamental facts about contact Riemannian manifolds and we review their associated CR-structures. In Section 3, we study the curvature tensor R of contact Riemannian manifolds of class $\mathfrak{Q}_1$. In Section 4, we obtain the nice form of the curvature tensor of a contact Riemannian manifold of class $\mathfrak{Q}$ and of constant ϕ -holomorphic sectional

curvature. We call a simply connected and complete space belonging to $\mathfrak{Q}$ and of constant ϕ -holomorphic sectional curvature a *contact Riemannian space form*. In case of dimension three, the contact Riemannian space forms coincide with contact (k, μ) -spaces (see Remark 4.5). In higher dimensions, we can find (non-Sasakian) contact Riemannian space forms of constant ϕ -holomorphic sectional curvature $H > 0$, $H = 0$, or $H < 0$. From this observation, we see that the contact Riemannian space form is a proper extension of the Sasakian space form. Furthermore, in Section 4 we prove the equivalence theorem, the homogeneity and the ϕ -symmetry of a contact Riemannian space form.

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2. Preliminaries

All manifolds in the present paper are assumed to be connected and of class C^∞ . A $(2n + 1)$ -dimensional manifold M^{2n+1} is said to be a contact manifold if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η , we have a unique vector field ξ , which is called the characteristic vector field, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X . It is well-known that there exists an associated Riemannian metric g and a $(1, 1)$ -type tensor field ϕ such that

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y), \quad \phi^2 X = -X + \eta(X)\xi, \quad (2.1)$$

where X and Y are vector fields on M . From (2.1) it follows that

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \quad (2.2)$$

Although g and ϕ are not uniquely determined for η , the two pairs (η, g) and (η, ϕ) are canonically related to each other by the equation $d\eta(X, Y) = g(X, \phi Y)$. A Riemannian manifold M equipped with structure tensors (η, g) satisfying (2.1) is said to be a contact Riemannian manifold and is denoted by $M = (M; \eta, g)$. Given a contact Riemannian manifold M ,

we define a $(1,1)$ -type tensor field h by $h = \frac{1}{2}L_\xi\phi$, where L denotes Lie differentiation. Then we may observe that h is symmetric and satisfies

$$h\xi = 0, \quad h\phi = -\phi h, \quad (2.3)$$

$$(\nabla_X\eta)(Y) = g(X, \phi Y) - g(X, \phi hY) \quad (\text{or } \nabla_X\xi = -\phi X - \phi hX), \quad (2.4)$$

$$\nabla_\xi h = \phi - \phi R_\xi - \phi h^2, \quad (2.5)$$

where ∇ is the Levi-Civita connection and $R_\xi = R(\cdot, \xi)\xi$. Here, R is the Riemannian curvature tensor of M defined by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]}Z$$

for all vector fields X, Y, Z on M .

A contact Riemannian manifold for which ξ is Killing, is called a K -contact Riemannian manifold. It is easy to see that a contact Riemannian manifold is K -contact if and only if $h = 0$. For a contact Riemannian manifold M , one may define naturally an almost complex structure J on $M \times \mathbb{R}$ by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

where X is a vector field tangent to M , t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, M is said to be normal or Sasakian. It is known that M is normal if and only if M satisfies

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ . A Sasakian manifold is characterized by a condition

$$(\nabla_X\phi)Y = g(X, Y)\xi - \eta(Y)X \quad (2.6)$$

for all vector fields X and Y on the manifold.

It is well-known that a contact Riemannian manifold M is Sasakian if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

for all vector fields X and Y . For more details about contact Riemannian manifolds, we refer to [5], [10], [11].

For a contact Riemannian manifold $M = (M; \eta, \phi)$, the tangent space $T_p M$ of M at each point $p \in M$ is decomposed as $T_p M = D_p \oplus \{\xi\}_p$ (direct sum), where we denote $D_p = \{v \in T_p M \mid \eta(v) = 0\}$. Then $D : p \rightarrow D_p$ defines a distribution orthogonal to ξ . The $2n$ -dimensional distribution D is called the *contact distribution*. We see that the restriction $\bar{\phi} = \phi|_D$ of ϕ to D defines an almost complex structure on D . Then the associated CR-structure of a given contact Riemannian manifold M is given by the holomorphic subbundle

$$\mathcal{H} = \{X - i\bar{\phi}X : X \in D\}$$

of the complexification $TM^{\mathbb{C}}$ of the tangent bundle TM . Then we see that each fiber $\mathcal{H}_x$ ($x \in M$) is of complex dimension n and $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$. Furthermore, we have $\mathbb{C}D = \mathcal{H} \oplus \bar{\mathcal{H}}$. We say that *the associated CR-structure is integrable* if $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$. For $\mathcal{H}$, we define the Levi form, by

$$L : D \times D \rightarrow \mathcal{F}(M), \quad L(X, Y) = -d\eta(X, \phi Y)$$

where $\mathcal{F}(M)$ denotes the algebra of differential functions on M . Then we see that the Levi form is hermitian and positive definite, that is, the CR-structure is a *strongly pseudo-convex, pseudo-hermitian CR-structure*.

Since $d\eta(\phi X, \phi Y) = d\eta(X, Y)$, we see that $[\bar{\phi}X, \bar{\phi}Y] - [X, Y] \in D$ and $[\phi X, Y] + [X, \phi Y] \in D$ for $X, Y \in D$. Thus, the associated CR-structure is integrable ($[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$) if and only if

$$[\bar{\phi}, \bar{\phi}](X, Y) = 0$$

for $X, Y \in D$, where $[\bar{\phi}, \bar{\phi}]$ is the Nijenhuis torsion of $\bar{\phi}$. It was obtained ([14, Proposition 2.1]) that for a given contact Riemannian manifold M the associated CR-structure is strongly pseudo-convex integrable if and only if M satisfies the integrability condition $Q_1 = 0$, where Q_1 is a $(1, 2)$ -type tensor field on M defined by

$$Q_1(X, Y) = (\nabla_X \phi)Y - g(X + hX, Y)\xi + \eta(Y)(X + hX)$$

for all vector fields X, Y on M . We remark here that the class of contact strongly pseudo-convex integrable CR-manifolds is invariant under D -homothetic deformations ([12])

$$\tilde{\eta} = a\eta, \quad \tilde{\xi} = \frac{1}{a}\xi, \quad \tilde{\phi} = \phi, \quad \tilde{g} = ag + a(a-1)\eta \otimes \eta,$$

where a is a positive constant. In fact, by direct computations, we have $\tilde{h} = \frac{1}{a}h$ and

$$(\tilde{\nabla}_X \tilde{\phi})Y = (\nabla_X \phi)Y + (a-1)\eta(Y)\phi^2 X - \frac{a-1}{a}g(X, hY)\xi.$$

From this, we easily see that $Q_1 = 0$ implies $\tilde{Q}_1 = 0$. Taking account of (2.6) we see that for a Sasakian manifold the associated CR-structure is strongly pseudo-convex integrable ($h = 0$). For 3-dimensional contact Riemannian manifolds their associated CR structures are always integrable (see [14]). One other class of contact Riemannian manifolds whose associated-CR structure is integrable is the class which is determined by the condition (see [6])

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

$k, \mu \in \mathbb{R}$. A space belonging to this class is called a contact (k, μ) -space ([3], [6]). In [6] the authors proved that $k \leq 1$. If $k = 1$, then $h = 0$ and the structure is Sasakian. It was also proved in [6] that the standard contact Riemannian structure of the unit tangent sphere bundle is a (k, μ) -space if and only if the base manifold is of constant curvature c with $k = c(2 - c)$ and $\mu = -2c$.

Now, we give

Definition 2.1. The class $\mathfrak{Q}_1$ is formed by the contact Riemannian manifolds whose associated CR-structures are integrable, that is,

$$\mathfrak{Q}_1 = \{(M, \eta, g) : Q_1 = 0\}.$$

3. A class $\mathfrak{Q}_1$ of contact Riemannian manifolds

Let M be a contact Riemannian manifold belonging to $\mathfrak{Q}_1$. Then we have

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX) \quad (3.1)$$

for all vector fields X and Y . Comparing with (2.6), we see that a contact Riemannian manifold $\in \mathfrak{Q}_1$ is normal (or Sasakian) if and only if $h = 0$. We put $P(X, Y) = (\nabla_X h)Y - (\nabla_Y h)X$. Then we see that P is a $(1, 2)$ -type

tensor field on M . We have the following

Proposition 3.1. *For all vector fields X, Y, Z on M*

$$R(X, Y)\xi = \eta(Y)(X + hX) - \eta(X)(Y + hY) + \phi P(Y, X), \quad (3.2)$$

$$\begin{aligned} g(R(\xi, X)Y, Z) &= \eta(Z)g(Y + hY, Z) - \eta(Y)g(Z + hZ, X) \\ &\quad + g(\phi P(Z, Y), X), \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} R(X, Y)\phi Z &= \phi R(X, Y)Z - g(Y + hY, Z)(\phi X + \phi hX) \\ &\quad + g(X + hX, Z)(\phi Y + \phi hY) + g(\phi X + \phi hX, Z)(Y + hY) \\ &\quad - g(\phi Y + \phi hY, Z)(X + hX) + g(P(X, Y), Z)\xi \\ &\quad - \eta(Z)P(X, Y). \end{aligned} \quad (3.4)$$

PROOF. From the definition of the curvature tensor R , by using (2.4), (3.1) and the fundamental symmetries of the curvature tensor, we obtain (3.2) and (3.3). The Ricci identity for ϕ is given as

$$R(X, Y)\phi Z - \phi R(X, Y)Z = (\nabla_{X,Y}^2 \phi)Z - (\nabla_{Y,X}^2 \phi)Z, \quad (3.5)$$

where $\nabla_{X,Y}^2 = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$. From (3.1) we have

$$\begin{aligned} (\nabla_{X,Y}^2 \phi)Z &= -g(Y + hY, Z)(\phi X + \phi hX) + g(\phi X + \phi hX, Z)(Y + hY) \\ &\quad + g((\nabla_X h)Y, Z)\xi - \eta(Z)(\nabla_X h)Y, \end{aligned} \quad (3.6)$$

and thus (3.4) follows easily from this and (3.5). $\square$

Now, we prove

Proposition 3.2. *Let M be a space $\in \mathfrak{Q}_1$. Then the necessary and sufficient condition for M to have pointwise constant ϕ -holomorphic sectional curvature H is*

$$\begin{aligned} &g(R(X, Y)Z, W) \\ &= \frac{1}{4} \{ (H + 3) [(g(Y, Z) - \eta(Y)\eta(Z))(g(X, W) - \eta(X)\eta(W)) \\ &\quad - (g(X, Z) - \eta(X)\eta(Z))(g(Y, W) - \eta(Y)\eta(W))] \} \end{aligned}$$

$$\begin{aligned}
& + (H-1) [g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) - 2g(\phi X, Y)g(\phi Z, W)] \\
& + 2 [2g(hY, Z)(g(X, W) - \eta(X)\eta(W)) - 2g(hX, Z)(g(Y, W) - \eta(Y)\eta(W)) \\
& + 2g(hX, W)(g(Y, Z) - \eta(Y)\eta(Z)) - 2g(hY, W)(g(X, Z) - \eta(X)\eta(Z)) \\
& + g(hY, Z)g(hX, W) - g(hX, Z)g(hY, W) - g(hY, \phi Z)g(hX, \phi W) \\
& + g(hX, \phi Z)g(hY, \phi W)] \} \\
& + \eta(X) [g(\phi P(W, Z) - \eta(W)\phi P(\xi, Z) - \eta(Z)\phi P(W, \xi), Y)] \\
& - \eta(Y) [g(\phi P(W, Z) - \eta(W)\phi P(\xi, Z) - \eta(Z)\phi P(W, \xi), X)] \\
& + \eta(Z) [g(\phi P(Y, X) - \eta(Y)\phi P(\xi, X) - \eta(X)\phi P(Y, \xi), W)] \\
& - \eta(W) [g(\phi P(Y, X) - \eta(Y)\phi P(\xi, X) - \eta(X)\phi P(Y, \xi), Z)] \\
& - \eta(X)\eta(Z) [g(Y + hY, W) - \eta(Y)\eta(W) + g(\phi P(\xi, Y), W)] \\
& + \eta(X)\eta(W) [g(Y + hY, Z) - \eta(Y)\eta(Z) + g(\phi P(\xi, Y), Z)] \\
& + \eta(Y)\eta(Z) [g(X + hX, W) - \eta(X)\eta(W) + g(\phi P(\xi, X), W)] \\
& - \eta(Y)\eta(W) [g(X + hX, Z) - \eta(X)\eta(Z) + g(\phi P(\xi, X), Z)] \tag{3.7}
\end{aligned}$$

for all vector fields X, Y, Z, W in M .

PROOF. For $X, Y \in D$, using the first Bianchi identity and the fundamental properties of the curvature tensor, (2.1), (2.2) and (2.3), we obtain from (3.4)

$$g(R(X, \phi X)Y, \phi Y) = g(R(X, \phi Y)Y, \phi X) + g(R(X, Y)\phi X, \phi Y) \tag{3.8}$$

and

$$\begin{aligned}
g(R(X, Y)\phi X, \phi Y) & = g(R(X, Y)X, Y) - g(X, Y)^2 - g(hX, Y)^2 \\
& - 2g(X, Y)g(hX, Y) + g(X, X)g(Y, Y) + g(X, X)g(hY, Y) \\
& + g(Y, Y)g(hX, X) + g(hX, X)g(hY, Y) - g(\phi X, Y)^2 \\
& + g(\phi hX, Y)^2 - g(\phi hX, X)g(\phi hY, Y). \tag{3.9}
\end{aligned}$$

Similarly, from (3.4) we get

$$g(R(X, \phi Y)X, \phi Y) = g(R(X, \phi Y)Y, \phi X) + g(X, Y)^2 - g(hX, Y)^2$$

$$\begin{aligned}
& -g(\phi hX, X)g(\phi hY, Y) - g(X, X)g(Y, Y) - g(Y, Y)g(hX, X) \\
& + g(X, X)g(hY, Y) + g(hX, X)g(hY, Y) + g(\phi X, Y)^2 \\
& + g(\phi hX, Y)^2 + 2g(\phi X, Y)g(\phi hX, Y)
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
g(R(Y, \phi X)Y, \phi X) &= g(R(X, \phi Y)Y, \phi X) + g(X, Y)^2 - g(hX, Y)^2 \\
& - g(\phi hX, X)g(\phi hY, Y) - g(X, X)g(Y, Y) + g(Y, Y)g(hX, X) \\
& - g(X, X)g(hY, Y) + g(hX, X)g(hY, Y) + g(\phi X, Y)^2 \\
& + g(\phi hX, Y)^2 - 2g(\phi X, Y)g(\phi hX, Y).
\end{aligned} \tag{3.11}$$

We now suppose that the ϕ -holomorphic sectional curvature is constant, i.e., $K(X, \phi X) = H$ for any $X \in D$. Then we have

$$g(R(X, \phi X)\phi X, X) = Hg(X, X)^2, \tag{3.12}$$

for any $X \in D$. Replacing X by $X + Y$ and $X - Y$ for $X, Y \in D$ in (3.12) respectively, and summing them, we get

$$\begin{aligned}
& 2g(R(X, \phi X)\phi Y, Y) + g(R(X, \phi Y)\phi Y, X) + 2g(R(X, \phi Y)\phi X, Y) \\
& + g(R(Y, \phi X)\phi X, Y) = 2H\{2g(X, Y)^2 + g(X, X)g(Y, Y)\}.
\end{aligned} \tag{3.13}$$

From (3.8), (3.9), (3.10), (3.11) and (3.13), we get

$$\begin{aligned}
& 3g(R(X, \phi Y)\phi X, Y) + g(R(X, Y)Y, X) + 2g(hX, Y)^2 \\
& + 2g(X, Y)g(hX, Y) - g(X, X)g(hY, Y) - g(Y, Y)g(hX, X) \\
& - 2g(hX, X)g(hY, Y) - 2g(\phi hX, Y)^2 + 2g(\phi hX, X)g(\phi hY, Y) \\
& = H\{2g(X, Y)^2 + g(X, X)g(Y, Y)\}.
\end{aligned} \tag{3.14}$$

Replacing Y by ϕY in (3.14) and using (2.1), (2.2) and (2.3), we have

$$\begin{aligned}
& 3g(R(X, Y)\phi Y, \phi X) - g(R(X, \phi Y)X, \phi Y) + 2g(\phi hX, Y)^2 \\
& - 2g(X, \phi Y)g(hX, \phi Y) + g(X, X)g(hY, Y) - g(Y, Y)g(hX, X) \\
& + 2g(hX, X)g(hY, Y) - 2g(hX, Y)^2 - 2g(\phi hX, X)g(\phi hY, Y)
\end{aligned}$$

$$= H\{2g(X, \phi Y)^2 + g(X, X)g(Y, Y)\}. \quad (3.15)$$

From (3.15), together with (3.9) and (3.10), we get

$$\begin{aligned} & -3g(R(X, Y)Y, X) - g(R(X, \phi Y)\phi X, Y) + 2g(X, Y)^2 + 2g(hX, Y)^2 \\ & + 6g(X, Y)g(hX, Y) - 2g(X, X)g(Y, Y) - 3g(X, X)g(hY, Y) \\ & - 3g(Y, Y)g(hX, X) - 2g(hX, X)g(hY, Y) \\ & + 2g(X, \phi Y)^2 - 2g(\phi hX, Y)^2 + 2g(\phi hX, Y)g(\phi hY, Y) \\ & = H\{2g(X, \phi Y)^2 + g(X, X)g(Y, Y)\}. \end{aligned} \quad (3.16)$$

From (3.14) and (3.16), we have

$$\begin{aligned} 4g(R(X, Y)Y, X) &= (H + 3)\{g(X, X)g(Y, Y) - g(X, Y)^2\} \\ &+ 3(H - 1)g(X, \phi Y)^2 - 2\{g(hX, Y)^2 + 4g(X, Y)g(hX, Y) \\ &- 2g(X, X)g(hY, Y) - 2g(Y, Y)g(hX, X) \\ &- g(hX, X)g(hY, Y) - g(\phi hX, Y)^2 + g(\phi hX, X)g(\phi hY, Y)\} \end{aligned} \quad (3.17)$$

for any $X, Y \in D$. Replacing $X = X + Z$ in (3.17), we obtain

$$\begin{aligned} 4g(R(X, Y)Y, Z) &= (H + 3)\{g(X, Z)g(Y, Y) - g(X, Y)g(Y, Z)\} \\ &+ 3(H - 1)g(X, \phi Y)g(Z, \phi Y) - 2\{g(hX, Y)g(hY, Z) \\ &+ 2g(X, Y)g(hY, Z) + 2g(Y, Z)g(hX, Y) \\ &- 2g(X, Z)g(hY, Y) - 2g(Y, Y)g(hX, Z) - g(hX, Z)g(hY, Y) \\ &- g(\phi hX, Y)g(\phi hZ, Y) + g(\phi hX, Z)g(\phi hY, Y)\}. \end{aligned} \quad (3.18)$$

If we replace $Y = Y + W$ in (3.18) again and use (2.3), then we obtain

$$\begin{aligned} & 4\{g(R(X, Y)W, Z) + g(R(X, W)Y, Z)\} \\ &= (H + 3)\{2g(X, Z)g(Y, W) - g(X, Y)g(W, Z) - g(X, W)g(Y, Z)\} \\ &+ 3(H - 1)\{g(X, \phi Y)g(Z, \phi W) + g(X, \phi W)g(Z, \phi Y)\} \\ &- 2\{g(hX, Y)g(hZ, W) + g(hX, W)g(hZ, Y) + 2g(X, Y)g(hZ, W) \\ &+ 2g(X, W)g(hZ, Y) + 2g(Z, Y)g(hX, W) + 2g(Z, W)g(hX, Y)\} \end{aligned}$$

$$\begin{aligned}
& -4g(X, Z)g(hY, W) - 4g(Y, W)g(hX, Z) - 2g(hX, Z)g(hY, W) \\
& -g(\phi hX, Y)g(\phi hZ, W) - g(\phi hX, W)g(\phi hZ, Y) \\
& + 2g(\phi hX, Z)g(\phi hY, W)\}
\end{aligned} \tag{3.19}$$

and we have

$$\begin{aligned}
& 4\{g(R(X, Z)W, Y) + g(R(X, W)Z, Y)\} \\
& = (H + 3)\{2g(X, Y)g(Z, W) - g(X, Z)g(W, Y) - g(X, W)g(Z, Y)\} \\
& + 3(H - 1)\{g(X, \phi Z)g(Y, \phi W) + g(X, \phi W)g(Y, \phi Z)\} \\
& - 2\{g(hX, Z)g(hY, W) + g(hX, W)g(hY, Z) \\
& + 2g(X, Z)g(hY, W) + 2g(X, W)g(hY, Z) + 2g(Y, Z)g(hX, W) \\
& + 2g(Y, W)g(hX, Z) - 4g(X, Y)g(hZ, W) - 4g(Z, W)g(hX, Y) \\
& - 2g(hX, Y)g(hZ, W) - g(\phi hX, Z)g(\phi hY, W) - g(\phi hX, W)g(\phi hY, Z) \\
& + 2g(\phi hX, Y)g(\phi hZ, W)\}.
\end{aligned} \tag{3.20}$$

We subtract (3.20) from (3.19). Then by using the first Bianchi identity and (2.3), we get

$$\begin{aligned}
4g(R(X, Y)Z, W) &= (H + 3)\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} + (H - 1) \\
& \times \{g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) - 2g(\phi X, Y)g(\phi Z, W)\} \\
& + 2\{2g(hY, Z)g(X, W) - 2g(hX, Z)g(Y, W) + 2g(Y, Z)g(hX, W) \\
& - 2g(X, Z)g(hY, W) + g(hY, Z)g(hX, W) - g(hX, Z)g(hY, W) \\
& - g(hY, \phi Z)g(hX, \phi W) + g(hX, \phi Z)g(hY, \phi W)\},
\end{aligned} \tag{3.21}$$

where $X, Y, Z, W \in D$. We now let X be an arbitrary vector field on M . Then we may write

$$X = X^T + \eta(X)\xi,$$

where X^T denotes the horizontal part of X . Then we have for all vector fields X, Y, Z, W in M :

$$g(R(X, Y)Z, W) = g(R(X^T, Y^T)Z^T, W^T) + \eta(X)g(R(\xi, Y^T)Z^T, W^T)$$

$$\begin{aligned}
& + \eta(Y)g(R(X^T, \xi)Z^T, W^T) + \eta(Z)g(R(X^T, Y^T)\xi, W^T) \\
& + \eta(W)g(R(X^T, Y^T)Z^T, \xi) + \eta(X)\eta(Z)g(R(\xi, Y^T)\xi, W^T) \\
& + \eta(X)\eta(W)g(R(\xi, Y^T)Z^T, \xi) + \eta(Y)\eta(Z)g(R(X^T, \xi)\xi, W^T) \\
& + \eta(Y)\eta(W)g(R(X^T, \xi)Z^T, \xi). \tag{3.22}
\end{aligned}$$

Furthermore, from (3.22), by using (3.2), (3.3), (3.21) and straightforward calculations, we obtain (3.7). $\square$

From (3.6), by using (2.4) and (2.5), we find for the Ricci tensors

$$\begin{aligned}
\rho(X, Y) = & \frac{1}{4} \{ (2(n+1)H + 6n - 2)(g(X, Y) - \eta(X)\eta(Y)) \\
& + 4(2n - 1)g(hX, Y) \} \\
& - \eta(X) \sum_i g(\phi P(Y, e_i), e_i) + \eta(Y) \sum_i g(\phi P(X, e_i), e_i) \\
& + g(\phi P(\xi, X), Y) + \eta(X)\eta(Y)(2n - \text{tr } h^2) \tag{3.23}
\end{aligned}$$

for all vector fields X and Y in M , where $\{e_i\}$ ($i = 1, 2, \dots, 2n+1$) is an arbitrary local orthonormal frame field on M . Since the trace of h vanishes, from (3.23), we have for the scalar curvature

$$\tau = \frac{1}{2} \cdot n(2(n+1)H + 6n - 2) + 2n - 2(\text{tr } h^2).$$

4. A class of contact Riemannian manifolds Ω

There are two typical examples of contact manifolds; one is formed by the principal circle bundles over symplectic manifolds of integral class (including the odd-dimensional spheres) and the other is given by the unit tangent sphere bundles. The former admit a Riemannian metric which is Sasakian. Concerning the latter, in [15], it was proved that the associated CR-structure of a unit tangent sphere bundle T_1M with standard contact Riemannian structure is integrable if and only if the base manifold is of constant curvature. Here, we note that the unit tangent sphere bundle of a space of constant curvature satisfies ([6])

$$g((\nabla_{X^T} h)Y^T, Z^T) = 0, \tag{1}$$

that is, h is η -parallel and at the same time it also satisfies

$$\nabla_\xi h = \mu h\phi \quad (2)$$

where μ is a constant. Now, we consider a contact Riemannian manifolds whose structure tensor h satisfies (1) and (2) simultaneously. Then

$$\begin{aligned} 0 &= g((\nabla_{X^T} h)Y^T, Z^T) = g((\nabla_{X-\eta(X)\xi} h)(Y - \eta(Y)\xi, Z - \eta(Z)\xi)) \\ &= g((\nabla_X h)Y, Z) - \eta(X)g((\nabla_\xi h)Y, Z) - \eta(Y)g((\nabla_X h)\xi, Z) \\ &\quad - \eta(Z)g((\nabla_X h)Y, \xi) + \eta(X)\eta(Y)g((\nabla_\xi h)\xi, Z) + \eta(Y)\eta(Z)g((\nabla_X h)\xi, \xi) \\ &\quad + \eta(Z)\eta(X)g((\nabla_\xi h)Y, \xi) - \eta(X)\eta(Y)\eta(Z)g((\nabla_\xi h)\xi, \xi). \end{aligned}$$

From the above equation, by using (2.3), (2.4) and $\nabla_\xi h = \mu h\phi$, we have

$$(\nabla_X h)Y = g((h - h^2)\phi X, Y)\xi + \eta(Y)(h - h^2)\phi X + \mu\eta(X)h\phi Y \quad (4.1)$$

for any vector fields X and Y . Now, we define a $(1, 2)$ -tensor field $Q_2(X, Y)$ by

$$\begin{aligned} Q_2(X, Y) &= (\nabla_X h)Y - g((h - h^2)\phi X, Y)\xi \\ &\quad - \eta(Y)(h - h^2)\phi X - \mu\eta(X)h\phi Y. \end{aligned}$$

Definition 4.1. The class $\mathfrak{Q}$ is given by the spaces belonging to $\mathfrak{Q}_1$ and satisfying $Q_2 = 0$, that is,

$$\mathfrak{Q} = \{(M, \eta, g) \in \mathfrak{Q}_1 : Q_2 = 0\}.$$

Then, as mentioned before, we can see that this class $\mathfrak{Q}$ is invariant under D -homothetic deformations. More precisely, for a D -homothetic deformation, we have

$$(\tilde{\nabla}_X h)Y = \tilde{g}((\tilde{h} - \tilde{h}^2)\tilde{\phi}X, Y)\tilde{\xi} + \tilde{\eta}(Y)(\tilde{h} - \tilde{h}^2)\tilde{\phi}X + \tilde{\mu}\tilde{\eta}(X)h\tilde{\phi}Y,$$

where $\tilde{\mu} = (2(a-1) + \mu)/a$. Furthermore, assume that $hY = \lambda Y$ for $Y \in D$ and $\|Y\| = 1$. Then from (4.1), we easily get $g((\nabla_X h)Y, Y) = 0$ for any vector X , from which we have

Lemma 4.2. *The eigenvalues of h are constant.*

Further, from (4.1), we have

$$P(X, Y) = -g((\phi h^2 + h^2 \phi)X, Y)\xi + \eta(X)((\mu - 1)h\phi Y + h^2 \phi Y) \quad (4.2) \\ - \eta(Y)((\mu - 1)h\phi X + h^2 \phi X),$$

$$\phi P(X, Y) = \eta(X)((\mu - 1)hY - h^2 Y) - \eta(Y)((\mu - 1)hX - h^2 X). \quad (4.3)$$

Now, we prove a Schur-type theorem for the class $\mathfrak{Q}$.

Theorem 4.3. *Let $M = (M^{2n+1}; \eta, g)$ ($n > 1$) be a contact Riemannian manifold belonging to the class $\mathfrak{Q}$. If the ϕ -holomorphic sectional curvature at any point of M is independent of the choice of ϕ -holomorphic section, then it is constant on M and the curvature tensor is given by*

$$g(R(X, Y)Z, W) = \frac{1}{4} \{ (c + 3) [(g(Y, Z) - \eta(Y)\eta(Z))(g(X, W) \\ - \eta(X)\eta(W)) - (g(X, Z) - \eta(X)\eta(Z))(g(Y, W) - \eta(Y)\eta(W))] \\ + (c - 1) [g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) - 2g(\phi X, Y)g(\phi Z, W)] \\ + 2[2g(hY, Z)(g(X, W) - \eta(X)\eta(W)) - 2g(hX, Z)(g(Y, W) - \eta(Y)\eta(W)) \\ + 2g(hX, W)(g(Y, Z) - \eta(Y)\eta(Z)) - 2g(hY, W)(g(X, Z) - \eta(X)\eta(Z)) \\ + g(hY, Z)g(hX, W) - g(hX, Z)g(hY, W) - g(hY, \phi Z)g(hX, \phi W) \\ + g(hX, \phi Z)g(hY, \phi W)] \} \\ - \eta(X)\eta(Z)g(Y + \mu hY - h^2 Y, W) + \eta(X)\eta(W)g(Y + \mu hY - h^2 Y, Z) \\ + \eta(Y)\eta(Z)g(X + \mu hX - h^2 X, W) \\ - \eta(Y)\eta(W)g(X + \mu hX - h^2 X, Z) \quad (4.4)$$

for all vector fields X, Y, Z, W in M .

PROOF. Suppose that M has pointwise constant ϕ -holomorphic sectional curvature H . Then, taking account of (4.1), (4.2) and (4.3), from (3.23) we obtain

$$\rho(X, Y) = \frac{1}{4} \{ (2(n + 1)H + 6n - 2)(g(X, Y) - \eta(X)\eta(Y)) \\ + 4(2n - 2 + \mu)g(hX, Y) - 4g(h^2 X, Y) \}$$

$$+ \eta(X)\eta(Y)(2n - \operatorname{tr} h^2), \quad (4.5)$$

$$\tau = \frac{1}{2} \cdot n \left(2(n+1)H + 6n - 2 \right) + 2n - 2(\operatorname{tr} h^2). \quad (4.6)$$

From (4.1) and by using (2.4) and Lemma 4.2, we have

$$\begin{aligned} (\nabla_X \rho)(Y, Z) &= \frac{1}{4} \{ (2(n+1)(XH))(g(Y, Z) - \eta(Y)\eta(Z)) \\ &\quad + (2(n+1)H + 6n - 2)(\eta(Z)g(\phi X + \phi hX, Y) \\ &\quad + \eta(Y)g(\phi X + \phi hX, Z)) \\ &\quad + 4(2n - 2 + \mu)g(g((h - h^2)\phi X, Y)\eta(Z) \\ &\quad + \eta(Y)g((h - h^2)\phi X, Z) + \mu\eta(X)g(h\phi Y, Z)) \\ &\quad - 4(g((h - h^2)\phi X, hY)\eta(Z) + g((h - h^2)\phi X, hZ)\eta(Y)) \} \\ &\quad + (\eta(Z)g(-\phi X - \phi hX, Y) \\ &\quad + \eta(Y)g(-\phi X - \phi hX, Z))(2n - \operatorname{tr} h^2), \end{aligned}$$

which yields

$$\sum_i (\nabla_{e_i} \rho)(X, e_i) = \frac{1}{2} (n+1) \{ (XH) - (\xi H)\eta(X) \}. \quad (4.7)$$

By the well-known formula

$$\nabla_X \tau = 2 \sum_i (\nabla_{e_i} \rho)(X, e_i)$$

for any local orthonormal frame field $\{e_i\}$ ($i = 1, 2, \dots, 2n+1$) and by using (4.6), (4.7) and Lemma 4.2, we have

$$(n+1) \{ XH - (\xi H)\eta(X) \} = n(n+1)XH.$$

This says that $\xi H = 0$ and $(n-1)XH = 0$. Since $n > 1$, we see that H is constant, say c . By applying (4.1), (4.2) and (4.3) in Proposition 3.2, we obtain (4.4). $\square$

Definition 4.4. A complete and simply connected contact Riemannian manifold of class $\mathfrak{Q}$ with constant ϕ -holomorphic sectional curvature is said to be a *contact Riemannian space form*.

So, from the proof for Proposition 3.2 and Theorem 4.3, we have

Theorem 4.5. *Let M be a complete and simply connected space belonging to the class $\mathfrak{Q}$. Then M is a contact Riemannian space form if and only if the curvature tensor R is given by (4.4).*

Remark 4.6. From (3.2) and (4.2), it follows that a manifold $\in \mathfrak{Q}$ satisfies

$$R(X, Y)\xi = \eta(Y)(X - h^2X) - \eta(X)(Y - h^2Y) + \mu(\eta(Y)hX - \eta(X)hY).$$

It is easily seen that the class $\mathfrak{Q}$ coincides with the class of contact (k, μ) -space if and only if $h^2 = (k - 1)\phi^2$, $k \in \mathbb{R}$. Thus, taking account of Lemma 4.2, we see that for dimension 3, the class $\mathfrak{Q}$ coincides with the class of contact (k, μ) -spaces. But, in higher dimensions, we do not know yet of an example in $\mathfrak{Q}$ which is not a contact (k, μ) -space.

Examples of (non-Sasakian) contact Riemannian space forms

(1) All 3-dimensional non-Sasakian contact (k, μ) -spaces have constant ϕ -holomorphic sectional curvatures $H = -(k + \mu)$. That is, $SU(2)$, the universal covering space of $SL(2, \mathbb{R})$, the universal covering space of the group $E(2)$ of rigid motions of Euclidean 2-space and the group space $E(1, 1)$ of rigid motions of Minkowski 2-space, respectively with a special left-invariant metric, are contact Riemannian space forms (see [6]).

(2) Tangent sphere bundles $T_1M(c)$ (with standard contact Riemannian structure) of n -dimensional spaces of constant curvature $c = 2 \pm \sqrt{5}$ have constant ϕ -holomorphic sectional $H = c^2$. We know that $T_1M(c)$ is simply connected when $n > 2$.

(3) By D -homothetic deformations, we can construct more examples with $H > 0$, $H < 0$, or $H = 0$. More explicitly, for the unit tangent sphere bundle of a space of constant curvature c ,

$$\begin{cases} H > 0, & \text{if } (-11 + 4\sqrt{6})/5 < c, \\ H = 0, & \text{if } c = (-11 + 4\sqrt{6})/5, \\ H < 0, & \text{if } -1 < c < (-11 + 4\sqrt{6})/5. \end{cases}$$

For the above (2) and (3), we refer to [9].

We close this section showing the equivalence theorem, the homogeneity and the ϕ -symmetry of the contact Riemannian space forms. First,

Theorem 4.7 (Equivalence theorem). *Let $(M^{2n+1}; \eta, g)$ and $(M^{2n+1}; \eta', g')$ be two contact Riemannian space forms with the same $\mu \in \mathbb{R}$. Suppose that the eigenvalues and the dimensions of their eigenspaces of h and h' are equal to each other. Then they are isometric as contact Riemannian spaces.*

PROOF. The theorem follows from the expression (4.4) for the Riemannian curvature tensor and the formulas (2.4), (3.1) and (4.1), using similar arguments as in [3] or [13]. $\square$

Now, we prove the homogeneity. We define a $(1, 2)$ -tensor field T by

$$\begin{aligned} T(X, Y) = & -g(\phi X + \phi hX, Y)\xi + \eta(Y)(\phi X + \phi hX) \\ & + (\mu/2)\eta(X)\phi Y \end{aligned} \quad (4.8)$$

for vector fields X, Y ([2]). Let $\bar{\nabla}$ be the connection determined by $\bar{\nabla} = \nabla + T$. Then we easily get

$$\bar{\nabla}g = 0, \quad \bar{\nabla}\xi = 0, \quad \bar{\nabla}\eta = 0, \quad \bar{\nabla}\phi = 0. \quad (4.9)$$

Also, we obtain from (4.1)

$$\bar{\nabla}h = 0. \quad (4.10)$$

Thus, in view of the form (4.4) and (4.8), using (4.9) and (4.10), it follows easily that

$$\bar{\nabla}T = 0 \quad (4.11)$$

and

$$\bar{\nabla}R = 0. \quad (4.12)$$

At last, together with (4.9), (4.11) and (4.12), by KIRIČENKO's generalization ([8]) of the Ambrose–Singer theorem ([1] or [16]), we have

Theorem 4.8 (Homogeneity). *A contact Riemannian space form is a locally homogeneous contact Riemannian manifold.*

We remark that a Sasakian space form is a naturally reductive homogeneous space ([7]). Furthermore, together with (2.4), (3.1), (4.1) and (4.4), and in a similar way as in the proof of Lemma 7 and Theorem 1 in [2], we obtain

Theorem 4.9 (ϕ -symmetry). *A contact Riemannian space form is locally ϕ -symmetric in the strong sense, that is, the characteristic reflections are local isometries.*

For more details about the ϕ -symmetry in the weak or strong sense, we refer to [2], [4].

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