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Gaussian measures on the affine group: uniqueness of embedding and supports

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Abstract. It is shown that a Gaussian measure on the affine group can be embedded only in a uniquely determined Gaussian semigroup. The starting point of the proof is the fact that a Gaussian Lévy process (i.e., a Lévy process with vanishing Lévy measure) on the affine group satisfies a certain stochastic differential equation. Moreover, we will give a complete description of supports of Gaussian measures on the affine group using Siebert's support formula.

1. Introduction

A probability measure μ on a locally compact topological group G is called *continuously embeddable* if there exists a convolution semigroup $(\mu_t)_{t\geqslant 0}$ of probability measures on G (i.e., $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \geqslant 0$, and $\lim_{t\downarrow 0} \mu_t = \mu_0 = \epsilon_e$) satisfying $\mu_1 = \mu$.

For general locally compact topological groups G one does not know whether the embedding convolution semigroup of a continuously embeddable probability measure on G is unique. If $(\mu_t)_{t\geqslant 0}$ and $(\nu_t)_{t\geqslant 0}$ are convolution semigroups of probability measures on $(\mathbb{R}^d, +)$ then it is well-known that $\mu_1 = \nu_1$ implies $\mu_t = \nu_t$ for all $t \geqslant 0$. The same statement holds for

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locally compact Abelian groups without non-trivial compact subgroups (cf. Heyer [7, Theorem 3.5.15]). For stable and semi-stable semigroups on simply connected nilpotent Lie groups see Drisch and Gallardo [2], Nobel [14] and a detailed discussion by Hazod and Siebert [6, Section 2.6]. Neuenschwander [12] studied Poisson semigroups on simply connected nilpotent Lie groups.

Pap [15] proved that a Gaussian measure on a simply connected nilpotent Lie group has a unique embedding semigroup among Gaussian convolution semigroups. We prove a similar result for the two-dimensional affine group (i.e., the group of proper affine mappings on the real line). Our method, which is related to the idea of Pap [15], consists of recursively calculating the first and second moments. In order to prove the uniqueness of embedding we consider a Gaussian Lévy process $(\xi(t))_{t\geqslant 0}$ in the affine group related to a Gaussian semigroup, and we show that $(\xi(t))_{t\geqslant 0}$ satisfies a certain stochastic differential equation (SDE). The question about the existence of a non-Gaussian embedding semigroup of a Gaussian measure remains still open. In the special case of simply connected step 2-nilpotent Lie groups Neuenschwander [13] showed that a Gaussian measure does not admit a non-Gaussian embedding semigroup.

We will also investigate the support of μ_t for $t > 0$ where $(\mu_t)_{t\geqslant 0}$ forms a Gaussian convolution semigroup on the affine group. Siebert [17] has shown that given a Gaussian semigroup $(\mu_t)_{t\geqslant 0}$ on a connected Lie group G, either the measures μ_t are absolutely continuous with respect to the Haar measure on G for all $t > 0$, or the measures μ_t are singular with respect to the Haar measure on G for all $t > 0$. In the first case we say that $(\mu_t)_{t\geqslant 0}$ is an absolutely continuous semigroup on G, otherwise it is called singular. For any absolutely continuous Gaussian semigroup $(\mu_t)_{t\geqslant 0}$ on a connected Abelian Lie group G, we have $\operatorname{supp}(\mu_t) = G$ for all $t > 0$, where $\operatorname{supp}(\mu)$ denotes the support of the measure μ. McCrudden [10] showed that for any absolutely continuous Gaussian semigroup $(\mu_t)_{t\geqslant 0}$ on any connected nilpotent Lie group G, we have $\operatorname{supp}(\mu_t) = G$ for all $t > 0$. But in the solvable case the situation becomes more complicated. Siebert [17] has shown that on the affine group there exists an absolutely continuous Gaussian semigroup $(\mu_t)_{t\geqslant 0}$ with $\operatorname{supp}(\mu_t) \neq G$ for every $t > 0$. We will give a complete description of supports for Gaussian semigroups on the affine group using Siebert's support formula. See further investigations on other Lie groups by McCrudden [9], [10], [11], Kelly-Lyth and McCrudden [8].

2. Gaussian Lévy processes

A Lévy process $(\xi(t))_{t\geqslant 0}$ in a locally compact topological group G is a stochastically continuous process with stationary independent left increments such that $\xi(0) = e$, where e denotes the unit element of G. A Gaussian Lévy process (it can also be called a Brownian motion) is a Lévy process with vanishing Lévy measure. For Lévy processes in locally compact topological groups see, e.g. Heyer [7].

Roynette [16] gave a recursive formula for constructing Gaussian Lévy processes in an arbitrary nilpotent Lie group by the help of a corresponding Gaussian Lévy process in the corresponding Lie algebra, that is, by some independent Wiener processes in $\mathbb{R}$. The formula involves Itô integrals and reflects the group law. In Feinsilver and Schott [3], [4] one can find an operator approach (applicable for other Lie groups and using limit theorems) in order to obtain similar explicit formulas. Applebaum and Kunita [1] studied Lévy processes $(\xi(t))_{t\geqslant 0}$ with values in a connected Lie group G. They have shown that for all twice continuously differentiable function $f : G \to \mathbb{R}$ the process $(f(\xi(t)))_{t\geqslant 0}$ satisfies a stochastic differential equation connected to the infinitesimal generator of the process $(\xi(t))_{t\geqslant 0}$.

In case of the affine group it turns out that a Gaussian Lévy process $(\xi(t))_{t\geqslant 0}$ can be constructed by the help of one standard Wiener process, or two independent standard Wiener processes. The formula involves again Itô integrals and reflects the group law as in the case of nilpotent Lie groups (see, e.g., Roynette [16]).

In what follows let G be the affine group. The group G can be realized as the matrix group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0,\, b \in \mathbb{R} \right\}.$$

Obviously G is a simply connected solvable Lie group.

The Lie algebra $\mathcal{G}$ of G can be realized as the matrix algebra

$$\mathcal{G} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}.$$

Consider the basis $\{e_1, e_2\}$ of $\mathcal{G}$ defined by

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then we have the commutation relation $[e_1, e_2] = e_2$.

Lévy processes with values in a Lie group can be given by their infinitesimal generators containing left invariant differential operators. If $f : G \to \mathbb{R}$ is continuously differentiable then, for every $X \in \mathcal{G}$, we can introduce the left invariant differential operator $\widetilde{X}$ defined by

$$\widetilde{X} f(g) := \lim_{t \to 0} \frac{f(g \exp(tX)) - f(g)}{t}, \qquad g \in G.$$

A Lévy process $(\xi(t))_{t \geqslant 0}$ is a Gaussian Lévy process in G if and only if its infinitesimal generator admits the form

$$\widetilde{N} = \sum_{i=1}^{2} a_i \widetilde{e}_i + \frac{1}{2} \sum_{i,j=1}^{2} b_{i,j} \widetilde{e}_i \widetilde{e}_j, \tag{2.1}$$

where $a_1, a_2 \in \mathbb{R}$ and $B = (b_{i,j})_{1 \leqslant i,j \leqslant 2}$ is a real symmetric positive semidefinite matrix. The infinitesimal generator $\widetilde{N}$ can be written in the form

$$\widetilde{N} = \widetilde{Y} + \frac{1}{2} \sum_{k=1}^{r} \widetilde{X}_k^2, \tag{2.2}$$

where

$$\widetilde{Y} = \sum_{i=1}^{2} a_i \widetilde{e}_i, \qquad \widetilde{X}_j = \sum_{i=1}^{2} \sigma_{i,j} \widetilde{e}_i, \quad 1 \leqslant j \leqslant r \leqslant 2,$$

where $\Sigma = (\sigma_{i,j})$ is a $2 \times r$ matrix such that $B = \Sigma \cdot \Sigma^\top$ and $\operatorname{rank} \Sigma = \operatorname{rank} B = r$.

Theorem 2.3. *Let $(\xi(t))_{t \geqslant 0}$ be a Gaussian Lévy process in the affine group G with infinitesimal generator* (2.1)*. Then*

$$\xi(t) = \begin{pmatrix} \mathsf{e}^{Z_1(t)} & \int_0^t \mathsf{e}^{Z_1(s)} \, \mathsf{d}(Z_2(s) + b_{1,2} s/2) \\ 0 & 1 \end{pmatrix}, \qquad t \geqslant 0,$$

where

$$Z_i(t) = a_i t + \sum_{j=1}^{r} \sigma_{i,j} W_j(t), \qquad i = 1, 2,$$

and $W_1(t)_{t\geqslant 0}$ and $W_2(t)_{t\geqslant 0}$ are independent standard Wiener processes.

Proof. Applying Theorem 3.1 in Applebaum and Kunita [1], $(\xi(t))_{t\geqslant 0}$ can be represented as a solution of the SDE

$$\begin{aligned}\xi(t) = I &+ \sum_{i=1}^{2} \int_0^t a_i \xi(s) e_i \, \mathsf{d}s + \frac{1}{2} \sum_{i,j=1}^{2} \int_0^t b_{i,j} \xi(s) e_i e_j \, \mathsf{d}s \\ &+ \sum_{i=1}^{2} \int_0^t \xi(s) e_i \, \mathsf{d}B_i(s),\end{aligned}$$

where I is the 2×2 identity matrix, and $B(t) = (B_1(t), B_2(t))$ is a Gaussian Lévy process in $\mathbb{R}^2$ with zero mean and covariances $\operatorname{cov}(B_i(t), B_j(t)) = tb_{i,j}$, $1 \leqslant i,\, j \leqslant 2$.

Writing $\xi(t)$ in the form

$$\xi(t) = \begin{pmatrix} \xi_1(t) & \xi_2(t) \\ 0 & 1 \end{pmatrix},$$

and using $e_1^2 = e_1$, $e_2^2 = 0$, $e_1 e_2 = e_2$, $e_2 e_1 = 0$ we obtain the SDE

$$\begin{aligned}\mathsf{d}\xi_1(t) &= \left(a_1 + \frac{b_{1,1}}{2}\right) \xi_1(t) \, \mathsf{d}t + \xi_1(t) \, \mathsf{d}B_1(t), \\ \mathsf{d}\xi_2(t) &= \left(a_2 + \frac{b_{1,2}}{2}\right) \xi_1(t) \, \mathsf{d}t + \xi_1(t) \, \mathsf{d}B_2(t).\end{aligned} \tag{2.4}$$

Clearly $B_1(t) = \sum_{j=1}^{r} \sigma_{1,j} W_j(t)$ and $B_2(t) = \sum_{j=1}^{r} \sigma_{2,j} W_j(t)$, where $W_1(t)_{t\geqslant 0}$ and $W_2(t)_{t\geqslant 0}$ are independent standard Wiener processes. By a simple application of Itô's formula we obtain

$$\xi_1(t) = \mathsf{e}^{Z_1(t)}.$$

Moreover

$$\xi_2(t) = \int_0^t \xi_1(s) \, \mathsf{d}\left(\left(a_2 + \frac{b_{1,2}}{2}\right) s + \sum_{j=1}^{r} \sigma_{2,j} W_j(s) \right)$$

$$= \int_0^t \mathsf{e}^{Z_1(s)} \, \mathsf{d}(Z_2(s) + b_{1,2}s/2).$$

Hence the assertion. □

Remark 2.5. The process $(Z_1(t), Z_2(t))_{t \geqslant 0}$ is a Gaussian Lévy process in $\mathbb{R}^2$ with infinitesimal generator

$$\sum_{i=1}^{2} a_i \partial_i + \frac{1}{2} \sum_{i,\,j=1}^{2} b_{i,j} \partial_i \partial_j,$$

i.e., replacing in (2.1) the differential operators $\widetilde{e}_1$ and $\widetilde{e}_2$ by ∂_1 and ∂_2, respectively.

3. Uniqueness of embedding

Theorem 3.1. *Let $(\mu_t)_{t \geqslant 0}$ and $(\nu_t)_{t \geqslant 0}$ be Gaussian convolution semigroups on the affine group. If $\mu_1 = \nu_1$ then we have $\mu_t = \nu_t$ for all $t \geqslant 0$. In other words, a Gaussian measure on the affine group can be embedded only in a uniquely determined Gaussian semigroup.*

Proof. It is sufficient to show that by the help of the measure μ_1 we can construct the whole Gaussian semigroup $(\mu_t)_{t \geqslant 0}$. A convolution semigroup is uniquely determined by its infinitesimal generator, hence it is sufficient to construct the infinitesimal generator of $(\mu_t)_{t \geqslant 0}$. Consider a Gaussian Lévy process $(\xi(t))_{t \geqslant 0}$ which corresponds to $(\mu_t)_{t \geqslant 0}$. We will show that the distribution of $\xi(1)$ uniquely determines the parameters $a_1, a_2, b_{1,1}, b_{1,2}$ and $b_{2,2}$ of the infinitesimal generator (2.1). It turns out that the knowledge of the expectation vector and covariance matrix of $\xi(1)$ uniquely defines these parameters.

First we calculate the expectation of $\xi(t)$. Taking the expectation of the integrated forms of the stochastic differential equations (2.4) we obtain

$$\mathsf{E}\xi_1(t) = 1 + \left(a_1 + \frac{b_{1,1}}{2}\right) \int_0^t \mathsf{E}\xi_1(s) \, \mathsf{d}s,$$

$$\mathsf{E}\xi_2(t) = \left(a_2 + \frac{b_{1,2}}{2}\right) \int_0^t \mathsf{E}\xi_1(s) \, \mathsf{d}s.$$

It follows that

$$\mathsf{E}\xi_1(t) = e^{\left(a_1+\frac{b_{1,1}}{2}\right)t}, \tag{3.2}$$

$$\mathsf{E}\xi_2(t) = \left(a_2 + \frac{b_{1,2}}{2}\right) \int_0^t e^{\left(a_1+\frac{b_{1,1}}{2}\right)s} \,\mathsf{d}s. \tag{3.3}$$

Using Itô's formula we have the following stochastic differential equations

$$\begin{aligned}
\mathsf{d}\xi_1^2(t) &= 2\xi_1(t)\,\mathsf{d}\xi_1(t) + \mathsf{d}[\xi_1,\xi_1]_t,\\
\mathsf{d}\xi_2^2(t) &= 2\xi_2(t)\,\mathsf{d}\xi_2(t) + \mathsf{d}[\xi_2,\xi_2]_t,\\
\mathsf{d}(\xi_1(t)\xi_2(t)) &= \xi_2(t)\,\mathsf{d}\xi_1(t) + \xi_1(t)\,\mathsf{d}\xi_2(t) + \mathsf{d}[\xi_1,\xi_2]_t,
\end{aligned}$$

where $[\,.\,,.\,]_t$ denotes the cross quadratic variation of continuous semimartingales.

Taking into account (2.4) and the facts that $B_i(t) = \sum_{j=1}^r \sigma_{i,j} W_j(t)$, $i = 1, 2$ and $B = \Sigma\Sigma^\top$ we obtain

$$\begin{aligned}
\mathsf{d}\xi_1^2(t) &= 2\xi_1(t)\,\mathsf{d}\xi_1(t) + b_{1,1}\xi_1^2(t)\,\mathsf{d}t,\\
\mathsf{d}\xi_2^2(t) &= 2\xi_2(t)\,\mathsf{d}\xi_2(t) + b_{2,2}\xi_1^2(t)\,\mathsf{d}t,\\
\mathsf{d}(\xi_1(t)\xi_2(t)) &= \xi_2(t)\,\mathsf{d}\xi_1(t) + \xi_1(t)\,\mathsf{d}\xi_2(t) + b_{1,2}\xi_1^2(t)\,\mathsf{d}t.
\end{aligned}$$

Taking the expectation of the integrated forms of these equations we get

$$\begin{aligned}
\mathsf{E}\xi_1^2(t) &= 1 + 2\,(a_1 + b_{1,1}) \int_0^t \mathsf{E}\xi_1^2(s)\,\mathsf{d}s,\\
\mathsf{E}\xi_2^2(t) &= b_{2,2} \int_0^t \mathsf{E}\xi_1^2(s)\,\mathsf{d}s + (2a_2 + b_{1,2}) \int_0^t \mathsf{E}(\xi_1(s)\xi_2(s))\,\mathsf{d}s,\\
\mathsf{E}(\xi_1(t)\xi_2(t)) &= \left(a_2 + \frac{3}{2}b_{1,2}\right) \int_0^t \mathsf{E}\xi_1^2(s)\,\mathsf{d}s\\
&\quad + \left(a_1 + \frac{b_{1,1}}{2}\right) \int_0^t \mathsf{E}(\xi_1(s)\xi_2(s))\,\mathsf{d}s.
\end{aligned}$$

It can be easily checked that the unique solutions of these equations are the following

$$\mathsf{E}\xi_1^2(t) = e^{2(a_1+b_{1,1})t}, \tag{3.4}$$

$$\mathsf{E}(\xi_1(t)\xi_2(t)) = \left(a_2 + \frac{3}{2}b_{1,2}\right) e^{\left(a_1+\frac{b_{1,1}}{2}\right)t} \int_0^t e^{(a_1+\frac{3}{2}b_{1,1})s}\,\mathrm{d}s, \tag{3.5}$$

$$\begin{aligned}\mathsf{E}\xi_2^2(t) &= c\int_0^t e^{(a_1+\frac{b_{1,1}}{2})s}\left(\int_0^s e^{(a_1+\frac{3}{2}b_{1,1})u}\,\mathrm{d}u\right)\mathrm{d}s\\ &\quad + b_{2,2}\int_0^t e^{2(a_1+b_{1,1})s}\,\mathrm{d}s,\end{aligned} \tag{3.6}$$

where $c = (2a_2 + b_{1,2})\left(a_2 + \frac{3}{2}b_{1,2}\right)$. Using (3.2) and (3.4) with $t = 1$ we have

$$\begin{cases} a_1 + \dfrac{b_{1,1}}{2} = \log \mathsf{E}\xi_1(1), \\ 2(a_1 + b_{1,1}) = \log \mathsf{E}\xi_1^2(1). \end{cases}$$

This system of linear equations has a unique solution for a_1 and $b_{1,1}$. Substituting a_1 and $b_{1,1}$ into (3.3) and (3.5) with $t = 1$ we obtain a system of linear equations for a_2 and $b_{1,2}$ which has again a unique solution. Equation (3.6) yields that $b_{2,2}$ is unique, too. So the infinitesimal generator of the Gaussian convolution semigroup $(\mu_t)_{t\geqslant 0}$ is uniquely determined by μ_1. $\square$

4. Supports of Gaussian measures

Let $(\mu_t)_{t\geqslant 0}$ be a Gaussian semigroup on the affine group G with infinitesimal generator $\widetilde{N}$. SIEBERT [17] showed that according to the structure of $\widetilde{N}$ we can distinguish four different types of Gaussian semigroups:

(i) $\widetilde{N} = \widetilde{Y} + \frac{1}{2}(\widetilde{X}_1^2 + \widetilde{X}_2^2)$ with X_1 and X_2 linearly independent. Then the semigroup is absolutely continuous, it has a strictly positive analytic density and $\mathrm{supp}(\mu_t) = G$ for all $t > 0$. Moreover $\mathrm{rank}(B) = 2$.

(ii) $\widetilde{N} = \widetilde{Y} + \frac{1}{2}\widetilde{X}_1^2$ with Y and X_1 linearly independent and $[X_1, e_2] \neq 0$. Then the semigroup is absolutely continuous. Moreover $\mathrm{rank}(B) = 1$.

(iii) $\widetilde{N} = \widetilde{Y} + \frac{1}{2}\widetilde{X}_1^2$ with Y and X_1 linearly independent and $[X_1, e_2] = 0$. Then the semigroup is singular. Moreover $\mathrm{rank}(B) = 1$.

(iv) $\widetilde{N} = \widetilde{Y} + \frac{1}{2}\widetilde{X}_1^2$ with Y and X_1 linearly dependent. Then the semigroup is singular and is supported by the proper closed subgroup $\exp(\mathbb{R}X_1)$. Moreover $\mathrm{rank}(B) = 1$.

Our aim is to determine the support of Gaussian semigroups of type (ii) and (iii). In special cases (when $\widetilde{N} = \widetilde{e}_2 + \widetilde{e}_1^2$ and $\widetilde{N} = \widetilde{e}_1 + \widetilde{e}_2^2$) SIEBERT [17] has already described it.

Let $\mathcal{M}$ denote the Lie subalgebra generated by $\{X_i : 1 \leqslant i \leqslant r\}$. We will use Siebert's support formula

$$\operatorname{supp}(\mu_t) = \overline{\bigcup_{n=1}^{\infty} \Big(M \exp \frac{tY}{n}\Big)^n} \quad \text{for all } t > 0,$$

where M is the analytic subgroup of G corresponding to $\mathcal{M}$. (See SIEBERT [17].)

Theorem 4.1. *Let $(\mu_t)_{t \geqslant 0}$ be a Gaussian semigroup on the affine group G with infinitesimal generator $\widetilde{N}$.*

(a) *If $\widetilde{N}$ is of type* (ii) *then for all $t > 0$, he measure μ_t is supported by*

$$\begin{cases} \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0,\, b \geqslant \dfrac{b_{2,1}}{b_{1,1}}(a-1) \right\} & \textit{if} \quad a_2 b_{1,1} - a_1 b_{2,1} > 0, \\ \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0,\, b \leqslant \dfrac{b_{2,1}}{b_{1,1}}(a-1) \right\} & \textit{if} \quad a_2 b_{1,1} - a_1 b_{2,1} < 0. \end{cases}$$

(b) *If $\widetilde{N}$ is of type* (iii) *then the measure μ_t is supported by* $\exp(ta_1e_1)\exp(\mathbb{R}e_2)$ *for all $t > 0$.*

PROOF. In both cases we have $r = 1$ and $\widetilde{N} = \widetilde{Y} + \frac{1}{2}\widetilde{X}_1^2$, where $Y = a_1e_1 + a_2e_2$ and $X_1 = \sigma_{1,1}e_1 + \sigma_{2,1}e_2$.

(a) Now $\sigma_{1,1}e_2 = [X_1, e_2] \neq 0$, and Y and X_1 are linearly independent, hence $a_1\sigma_{2,1} - a_2\sigma_{1,1} \neq 0$, which implies $a_1 b_{2,1} - a_2 b_{1,1} \neq 0$.

First consider the case $a_1 = 0$. By induction,

$$\begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}^k = \begin{pmatrix} \alpha^k & \alpha^{k-1}\beta \\ 0 & 0 \end{pmatrix}, \qquad k = 1, 2, \ldots,$$

hence

$$\exp\left\{\begin{pmatrix}\alpha & \beta\\ 0 & 0\end{pmatrix}\right\} = \begin{cases}\begin{pmatrix}\mathsf{e}^{\alpha} & \beta\cdot\dfrac{\mathsf{e}^{\alpha}-1}{\alpha}\\ 0 & 1\end{pmatrix}, & \text{for } \alpha\neq 0,\\ \begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}, & \text{for } \alpha = 0.\end{cases}$$

Using this formula it can be easily checked by induction that the elements of the set $\big(M\exp\frac{tY}{n}\big)^n$ have the form $S=(s_{i,j})_{1\leqslant i,j\leqslant 2}$, where

$$\begin{cases} s_{1,1} = \mathsf{e}^{(\alpha_1+\cdots+\alpha_n)\sigma_{1,1}},\\ s_{1,2} = \dfrac{t}{n}a_2\mathsf{e}^{(\alpha_1+\cdots+\alpha_n)\sigma_{1,1}} + \dfrac{\sigma_{2,1}}{\sigma_{1,1}}\left(\mathsf{e}^{(\alpha_1+\cdots+\alpha_n)\sigma_{1,1}}-1\right)\\ \qquad\quad + \dfrac{t}{n}a_2\left(\mathsf{e}^{\alpha_1\sigma_{1,1}}+\cdots+\mathsf{e}^{(\alpha_1+\cdots+\alpha_{n-1})\sigma_{1,1}}\right),\\ s_{2,1} = 0,\\ s_{2,2} = 1,\end{cases}$$

and $\alpha_1,\ldots,\alpha_n\in\mathbb{R}$, $n\in\mathbb{N}$ can be arbitrary. The term $\mathsf{e}^{\alpha_1\sigma_{1,1}}+\cdots+\mathsf{e}^{(\alpha_1+\cdots+\alpha_{n-1})\sigma_{1,1}}$ attends every positive number. Hence $s_{1,2}\geqslant\frac{t}{n}a_2s_{1,1}+\frac{\sigma_{2,1}}{\sigma_{1,1}}(s_{1,1}-1)$ if $a_2>0$, and $s_{1,2}\leqslant\frac{t}{n}a_2s_{1,1}+\frac{\sigma_{2,1}}{\sigma_{1,1}}(s_{1,1}-1)$ if $a_2<0$. Using Siebert's supports formula and the facts that $\frac{\sigma_{2,1}}{\sigma_{1,1}}=\frac{b_{2,1}}{b_{1,1}}$ and $b_{1,1}>0$ we obtain the assertion.

If $a_1\neq 0$ then again by induction we obtain that the elements of the set $\big(M\exp\frac{tY}{n}\big)^n$ have the form $S=(s_{i,j})_{1\leqslant i,j\leqslant 2}$, where

$$\begin{cases} s_{1,1} = \mathsf{e}^{(\alpha_1+\cdots+\alpha_n)\sigma_{1,1}+ta_1},\\ s_{1,2} = \left(a_2\dfrac{1-\mathsf{e}^{-ta_1/n}}{a_1}+\dfrac{\sigma_{2,1}}{\sigma_{1,1}}\mathsf{e}^{-ta_1/n}\right)\mathsf{e}^{(\alpha_1+\cdots+\alpha_n)\sigma_{1,1}+ta_1} - \dfrac{\sigma_{2,1}}{\sigma_{1,1}}\\ \qquad\quad + \dfrac{\mathsf{e}^{ta_1/n}-1}{a_1}\left(a_2-\dfrac{\sigma_{2,1}}{\sigma_{1,1}}a_1\right)\\ \qquad\qquad \times\left(\mathsf{e}^{\alpha_1\sigma_{1,1}}+\cdots+\mathsf{e}^{(\alpha_1+\cdots+\alpha_{n-1})\sigma_{1,1}+(n-2)ta_1/n}\right),\\ s_{2,1} = 0,\\ s_{2,2} = 1,\end{cases}$$

and $\alpha_1,\ldots,\alpha_n\in\mathbb{R}$, $n\in\mathbb{N}$ can be arbitrary. The term $\mathsf{e}^{\alpha_1\sigma_{1,1}}+\cdots+e^{(\alpha_1+\cdots+\alpha_{n-1})\sigma_{1,1}+(n-2)ta_1/n}$ attends every positive number. Using the fact that $\frac{\mathsf{e}^{ta_1/n}-1}{a_1} > 0$ we have

$$s_{1,2} \geqslant \left(a_2 \frac{1-\mathsf{e}^{-ta_1/n}}{a_1} + \frac{\sigma_{2,1}}{\sigma_{1,1}} \mathsf{e}^{-ta_1/n} \right) s_{1,1} - \frac{\sigma_{2,1}}{\sigma_{1,1}} \quad \text{if } a_2 b_{1,1} - a_1 b_{2,1} > 0,$$

$$s_{1,2} \leqslant \left(a_2 \frac{1-\mathsf{e}^{-ta_1/n}}{a_1} + \frac{\sigma_{2,1}}{\sigma_{1,1}} \mathsf{e}^{-ta_1/n} \right) s_{1,1} - \frac{\sigma_{2,1}}{\sigma_{1,1}} \quad \text{if } a_2 b_{1,1} - a_1 b_{2,1} < 0.$$

Since

$$a_2 \frac{1-\mathsf{e}^{-ta_1/n}}{a_1} + \frac{\sigma_{2,1}}{\sigma_{1,1}} \mathsf{e}^{-ta_1/n} > \frac{\sigma_{2,1}}{\sigma_{1,1}} \quad \text{if } a_2 b_{1,1} - a_1 b_{2,1} > 0,$$

$$a_2 \frac{1-\mathsf{e}^{-ta_1/n}}{a_1} + \frac{\sigma_{2,1}}{\sigma_{1,1}} \mathsf{e}^{-ta_1/n} < \frac{\sigma_{2,1}}{\sigma_{1,1}} \quad \text{if } a_2 b_{1,1} - a_1 b_{2,1} < 0,$$

and

$$\lim_{n\to\infty} \frac{\mathsf{e}^{ta_1/n}-1}{a_1} = 0,$$

we get the assertion.

(b) Now $\sigma_{1,1} e_2 = [X_1, e_2] = 0$. Moreover Y and X_1 are linearly independent, hence $a_1\sigma_{2,1} - a_2\sigma_{1,1} \neq 0$, which implies $a_1 \neq 0$. The elements of the set $\big(M \exp \frac{tY}{n}\big)^n$ have the form

$$\begin{pmatrix} \mathsf{e}^{ta_1} & \frac{a_2}{a_1}(\mathsf{e}^{ta_1}-1) + \sigma_{2,1}\big(\alpha_1 + \alpha_2 \mathsf{e}^{ta_1/n} + \cdots + (\alpha_1 + \ldots + \alpha_n)\mathsf{e}^{(n-1)a_1 t/n}\big) \\ 0 & 1 \end{pmatrix},$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Using Siebert's support formula we get

$$\operatorname{supp}(\mu_t) = \left\{ \begin{pmatrix} \mathsf{e}^{ta_1} & \beta \\ 0 & 1 \end{pmatrix} : \beta \in \mathbb{R} \right\} \quad \text{for all } t > 0,$$

that is, $\operatorname{supp}(\mu_t) = \exp(ta_1 e_1 + \mathbb{R}e_2) = \exp(ta_1 e_1)\exp(\mathbb{R}e_2)$ for all $t > 0$. $\square$

Remark 4.2. In case (ii) the semigroup $(\mu_t)_{t\geqslant 0}$ is absolutely continuous and $\operatorname{supp}(\mu_t)$ is the same closed subsemigroup of G for all $t > 0$. In case (iii) the semigroup $(\mu_t)_{t\geqslant 0}$ is singular and $\operatorname{supp}(\mu_t)$ is a proper coset of the same closed normal subgroup $\exp(\mathbb{R}e_2)$ for all $t > 0$.

Remark 4.3. Let $(\xi_t)_{t\geqslant 0}$ be the Gaussian Lévy process in the affine group G with infinitesimal generator $\widetilde{N}$ of type (iii), i.e., $\widetilde{N} = a_1\widetilde{e}_1 + a_2\widetilde{e}_2 + \frac{1}{2}\sigma_{2,1}^2\widetilde{e}_2^2$, where $a_1 \neq 0$ and $\sigma_{2,1} \neq 0$. By Theorem 4.1, the distribution of $\xi(t)$ is singular for all $t > 0$. Since $a_1 \neq 0$, the distribution of $\xi(t)$ is not symmetric for all $t > 0$. But

$$\xi(t) = \eta\left(\frac{\mathsf{e}^{2a_1t}-1}{2a_1}\right)x(t), \qquad t \geqslant 0,$$

where

$$x(t) := \exp(a_1te_1 + a_2te_2) = \begin{pmatrix} \mathsf{e}^{a_1t} & a_2\frac{\mathsf{e}^{a_1t}-1}{a_1} \\ 0 & 1 \end{pmatrix},$$

and $(\eta(t))_{t\geqslant 0}$ is a symmetric Gaussian Lévy process with infinitesimal generator $\frac{1}{2}\sigma_{2,1}^2\widetilde{e}_2^2$. Indeed, by Theorem 2.3

$$\xi(t) = \begin{pmatrix} \mathsf{e}^{a_1t} & \int_0^t \mathsf{e}^{a_1s}\,\mathsf{d}(a_2s + \sigma_{2,1}W(s)) \\ 0 & 1 \end{pmatrix},$$

$$\eta(t) = \begin{pmatrix} 1 & \sigma_{2,1}\widetilde{W}(t) \\ 0 & 1 \end{pmatrix}, \quad t \geqslant 0,$$

where $(W(t))_{t\geqslant 0}$ and $(\widetilde{W}(t))_{t\geqslant 0}$ are standard Wiener processes. Clearly

$$\eta\left(\frac{\mathsf{e}^{2a_1t}-1}{2a_1}\right)x(t) = \begin{pmatrix} \mathsf{e}^{a_1t} & a_2\frac{\mathsf{e}^{a_1t}-1}{a_1} + \sigma_{2,1}\widetilde{W}\left(\frac{\mathsf{e}^{2a_1t}-1}{2a_1}\right) \\ 0 & 1 \end{pmatrix}, \quad t \geqslant 0.$$

Both processes

$$\left(\int_0^t \mathsf{e}^{a_1s}\,\mathsf{d}(a_2s + \sigma_{2,1}W(s))\right)_{t\geqslant 0}, \left(a_2\frac{\mathsf{e}^{a_1t}-1}{a_1} + \sigma_{2,1}\widetilde{W}\left(\frac{\mathsf{e}^{2a_1t}-1}{2a_1}\right)\right)_{t\geqslant 0}$$

are processes with independent increments in $\mathbb{R}$ starting from 0 and their increments on the interval $[s,t] \subset \mathbb{R}_+$ have a normal distribution with mean $a_2\frac{\mathsf{e}^{a_1t}-\mathsf{e}^{a_1s}}{a_1}$ and variance $\sigma_{2,1}^2\frac{\mathsf{e}^{2a_1t}-\mathsf{e}^{2a_1s}}{2a_1}$, hence the assertion. The process $(\eta(t))_{t\geqslant 0}$ can be considered as the symmetric counterpart of process $(\xi(t))_{t\geqslant 0}$. In fact $(x(t))_{t\geqslant 0}$ is a deterministic Lévy process on the affine group G, which can be considered as the shift part of the process $(\xi(t))_{t\geqslant 0}$. We note that using Totter's formula of Hazod [5], Siebert [17] showed that the distribution of $\xi(t)$ and $\eta\left(\frac{\mathrm{e}^{2a_1t}-1}{2a_1}\right)x(t)$ coincide for all $t \geqslant 0$ in the special case $a_1 = 1$, $a_2 = 0$ and $\sigma_{2,1} = 2$.

Moreover, it can be checked that if the Gaussian Lévy process $(\xi(t))_{t\geqslant 0}$ is of type different from (iii) then the decomposition $\xi(t) = \eta\big(c(t)\big)x(t)$, $t \geqslant 0$ does not hold with any function $c : \mathbb{R}_+ \to \mathbb{R}_+$.

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