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Integrable solutions of a functional equation related to Wilson's equation

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Dedicated to Professor Zenon Moszner on his seventieth birthday and to Professor Lajos Tamássy on his eightieth birthday

Abstract. We look for solutions (f, g) of equation (1) on $\mathbb{R}$ in the case where f is locally integrable and g is continuous at the origin. In particular, among solutions exponential functions show up. The study is motivated by E. Wachnicki's paper [7] dealing with an integral mean value theorem.

1. Introduction

We consider the functional equation

$$af(x) + bf(y) = f(ax + by)g(y - x), \qquad x, y \in \mathbb{R}, \tag{1}$$

where a, b are some positive reals, $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, are unknown functions.

On putting $a = b = \frac{1}{2}$ in (1) we arrive at

$$f(x) + f(y) = 2f\left(\frac{x+y}{2}\right)g(y - x), \qquad x, y \in \mathbb{R}, \tag{2}$$

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which has appeared in E. Wachnicki's paper [7] in connection with an integral mean value theorem. Equation (2) reduces to Wilson's one which is dealt with in J. Aczél's monograph [1, pp. 165–171], cf. also [6]. Results on other generalization of (2) are found in [8].

In this paper we shall determine (in Sections 5–7) the solutions (f, g) of equation (1) belonging to the class $\mathcal{F} \times \mathcal{G}$ of functions, where

$$\mathcal{F} := \{f : \mathbb{R} \to \mathbb{R},\ f \neq 0 \text{ and } f \text{ is integrable on } \mathbb{R}\},$$

and

$$\mathcal{G} := \{g : \mathbb{R} \to \mathbb{R},\ g \text{ is continuous at the origin}\}.$$

2. Preliminaries

Let us first observe that, putting $x = y$ in (1), we get

$$(a + b)f(x) = g(0)f((a + b)x), \qquad x \in \mathbb{R}, \tag{3}$$

whence $f(0)(a + b - g(0)) = 0$. In the case where $g(0) \neq 0$ equation (3) becomes the simple Schröder's functional equation (for $x \in \mathbb{R}$)

$$f(px) = qf(x), \tag{4}$$

where $p = a + b$, $q = (a + b)/g(0)$.

The following facts on solutions of (4) are either found in [2] or they can be easily derived from the theory of the Schröder equation presented in [3], cf. also [5] (in particular, Theorem 6.1, p. 137 in [3]).

Lemma 1. *Let I be an interval containing zero and assume that*

$$0 < p < 1. \tag{P}$$

(A) *If $|q| > 1$ then the only solution $f : I \to \mathbb{R}$ of* (4) *which is continuous at zero is the zero function, $f(x) = 0$ for $x \in I$.*

(B) *If*

$$q = p, \tag{5}$$

then every C^1-solution of (4) *in I is given by*

$$f(x) = \alpha x, \qquad x \in I, \tag{6}$$

where $\alpha \in \mathbb{R}$ is a constant.

In the case where $g(x) = 1$ for $x \in \mathbb{R}$, equation (1) takes the form

$$af(x) + bf(y) = f(ax + by), \qquad x, y \in \mathbb{R}, \tag{7}$$

which is a special case of the equation considered in M. KUCZMA's monograph [4]. From the Theorem 13.10.2 found there on p. 341 we obtain the following.

Proposition. *If a non-constant (Lebesgue) measurable function $f : \mathbb{R} \to \mathbb{R}$ fulfils equation (7) then there exist real numbers $\alpha \neq 0$ and β such that*

$$f(x) = \alpha x + \beta, \qquad x \in \mathbb{R},$$

and $\beta = 0$ when $a + b \neq 1$.

3. Case $a + b \neq 1$, constant solution f

We start with listing the cases in which f satisfying (1) is necessarily the constant function. If the constant is zero, then (1) is satisfied by any function g. We assume that

$$\text{(I)} \qquad p := a + b \neq 1 \quad (a > 0,\ b > 0).$$

Theorem 1. *Assume* (**I**). *The solutions (f, g) of equation* (1), *defined on $\mathbb{R}$, are the following:*

(i) *If $g(0) = 0$, then*

$$f(x) = 0,\ x \in \mathbb{R}, \quad g : \mathbb{R} \to \mathbb{R} \text{ is an arbitary function.} \tag{8}$$

(ii) *If $g(0) = p$ and f is continuous at zero, then either $c := f(0) \neq 0$ and*

$$f(x) = c, \qquad g(x) = p, \qquad x \in \mathbb{R},$$

or $c = 0$ and (8) *holds.*

(iii) *If $0 < |g(0)| < p < 1$ or $|g(0)| > p > 1$, and f is continuous at zero, then* (8) *holds.*

PROOF. According to the introductory remark in Section 2 we may concentrate on determining f satisfying (3).

(i) From (3) (which now reads $pf(x) = 0$ for $x \in \mathbb{R}$) we get $f = 0$.

(ii) Equation (3) (now $f(x) = f(px)$) yields

$$f(x) = f(p^n x), \quad n \in \mathbb{N},\ x \in \mathbb{R}. \tag{9}$$

Thus, if $0 < p < 1$ and f is continuous at zero, on letting $n \to \infty$, we get $f(x) = f(0) =: c$ for $x \in \mathbb{R}$. Hence (1) and (**I**) yield $cp = cg(y - x)$, $x, y \in \mathbb{R}$. We see that $g(t) = p$ for $t \in \mathbb{R}$ if $c \neq 0$, whereas g is arbitrary if $c = 0$. In the case where $p > 1$ we rewrite (9) in the form

$$f(p^{-n}x) = f(x), \quad n \in \mathbb{N},\ x \in \mathbb{R},$$

and argue as above to get the same conclusion. The assertions of (ii) are proved.

(iii) In view of (3), f satisfies in $\mathbb{R}$ equation (4) with $q := p/g(0)$. If $0 < |g(0)| < p < 1$, then $|q| > 1$ and condition (P) holds, so that from Lemma 1(A) we get $f(x) = 0$ for $x \in \mathbb{R}$. When $|g(0)| > p > 1$ we write equation (4) in the form

$$f\left(\frac{1}{p}x\right) = \frac{1}{q}f(x), \qquad x \in \mathbb{R}.$$

Because of $0 < \frac{1}{p} < 1$ and $\left|\frac{1}{q}\right| > 1$ Lemma 1(A) again works, yielding $f = 0$. $\square$

Remark 1. In the cases: $p < |g(0)| < 1$ or $p > |g(0)| > 1$ the solution f of (4) which is continuous in a neighborhood of the origin depends on an arbitrary function (cf. [1], and also [5, Theorem 3.1.3, p. 99]) and there is no way of finding solutions of (1) among them. In these cases necessarily $f(0) = 0$ (observe that since $f(p^n x) = q^n f(x)$, $n \in \mathbb{N},\ x \in \mathbb{R}$, the conditions $p < 1$ and $|q| < 1$ lead to a contradiction when $f(0) \neq 0$).

4. Regularity of solutions from the class $\mathcal{F} \times \mathcal{G}$

We shall prove the following.

Lemma 2. *If $a > 0$, $b > 0$ and $(f, g) \in \mathcal{F} \times \mathcal{G}$ is a solution to* (1), *then*

(a) $f \in C^\infty(\mathbb{R})$,

(b) $g \in C^\infty(J)$, *where J is an open interval containing zero, and* $g'(0) = 0$.

Proof. We put again $p := a + b$ (> 0).

(a) For an arbitrarily fixed $x_0 > 0$ let $\delta > 0$ be such that f is integrable in the interval $V = V(x_0) := [x_0 - p\delta, x_0 + p\delta]$.

Let $t \in (0, \delta)$ and $x \in V$. Replacing x in (1) by $x - bt$ and y by $x + at$ we get

$$af(x - bt) + bf(x + at) = f(px)g(pt), \qquad x \in V,\ t \in (0, \delta).$$

Since $f \in \mathcal{F}$ is not the zero function, we have $g(0) \neq 0$ (cf. Theorem 1 (i)) and f satisfies (4) (with $q = p/g(0)$), whence

$$af(x - bt) + bf(x + at) = qf(x)g(pt), \quad x \in V,\ t \in (0, \delta). \tag{10}$$

In particular, the function $(0, \delta) \ni t \mapsto g(pt)$ is integrable. We integrate (10) with respect to t over the interval $[0, \delta]$ to get

$$a\int_0^\delta f(x - bt)dt + b\int_0^\delta f(x + at)dt = qf(x)\int_0^\delta g(pt)dt. \tag{11}$$

After the substitutions $s = x - bt$ and $s = x + at$ in the respective integrals equation (11) turns over

$$k(x) = cqf(x), \qquad x \in V, \tag{12}$$

where

$$k(x) := \frac{a}{b}\int_{x-b\delta}^{x} f(s)ds + \frac{b}{a}\int_x^{x+a\delta} f(s)ds, \qquad x \in V, \tag{13}$$

and

$$c := \int_0^\delta g(pt)dt.$$

Since f is integrable, (13) says that k is continuous in V. In turn, (12) implies that f is continuous in V, yielding, again by (13), the differentiability of k on V, etc. Therefore the function f is of class C^∞ on V, i.e., as x_0 was arbitrary in $\mathbb{R}$, $f \in C^\infty(\mathbb{R})$.

(b) By letting

$$u = ax + by, v = y - x, \qquad x, y \in \mathbb{R},$$

we see that (1) is equivalent to:

$$af\left(\frac{u - bv}{p}\right) + bf\left(\frac{u + av}{p}\right) = f(u)g(v), \qquad u, v \in \mathbb{R}. \tag{14}$$

Since f is not identically zero and it is of class C^∞ on $\mathbb{R}$, there is an open interval containing zero, say J, on which g is of class C^∞.

Taking derivatives in (14) with respect to v we obtain

$$\frac{-ab}{p}f'\left(\frac{u - bv}{p}\right) + \frac{ab}{p}f'\left(\frac{u + av}{p}\right) = f(u)g'(v), \qquad u \in \mathbb{R},\ v \in J.$$

Letting $v = 0$ here we get the equality $f(u)\ g'(0) = 0$, whence, as $f \neq 0$, we have $g'(0) = 0$. □

5. Case $a + b \neq 1$

In this case if $f \in \mathcal{F}$ has a non-zero derivative at zero then it is a linear function, and $g \in \mathcal{G}$ is a constant function.

Theorem 2. *Assume* (**I**). *If* $(f, g) \in \mathcal{F} \times \mathcal{G}$ *is a solution to* (1), f *is differentiable at zero, and*

$$f(0) = 0, \quad f'(0) \neq 0, \tag{15}$$

then

$$f(x) = \alpha x; \quad g(x) = 1, \qquad x \in \mathbb{R}, \tag{16}$$

where $\alpha \neq 0$ *is a real number.*

PROOF. Assume (15). The function f satisfies equation (3), i.e.,

$$pf(x) = g(0)f(px), \quad x \in \mathbb{R}, \tag{17}$$

whence

$$p\frac{f(x)}{x} = g(0)p\frac{f(px)}{px}, \quad x \in \mathbb{R} \setminus \{0\}.$$

Since $f(0) = 0$ and f is differentiable at zero, we get $pf'(0)(g(0) - 1) = 0$ and by (**I**) and (15) we have

$$g(0) = 1. \tag{18}$$

Because of (18) the SCHRÖDER equation (17) for f becomes

$$f(px) = pf(x), \qquad x \in \mathbb{R}, \tag{19}$$

and, by Lemma 2 (a), f is of class $C^\infty(\mathbb{R})$. If $p < 1$ then conditions (P) and (5) are satisfied, Lemma 1 (B) works, and we get formula (6) (i.e., (16) for $f(x)$), whereas if $p > 1$ it is enough to write (19) in the form

$$f(p^{-1}x) = p^{-1}\, f(x), \qquad x \in \mathbb{R},$$

to have Lemma 1 (B) applicable again. Using (6) in (1) we get $g(x) = 1$ and formula (16) holds true. □

6. Case $a + b = 1$; a differential equation

We pass to the remaining case

(**II**) $$p := a + b = 1 \quad \text{and} \quad 0 < a < b.$$

(Solutions of equation (1) when $a = b = 1/2$ are described in the paper [6].)

We have the following

Lemma 3. *Assume* (**II**). *If* $(f, g) \in \mathcal{F} \times \mathcal{G}$ *is a solution to* (1), *then*

(a) *the function* g *fulfils condition* (18),

(b) *the function* f *satisfies the differential equation*

$$abf''(x) = f(x)g''(0), \qquad x \in \mathbb{R}. \tag{20}$$

Proof. (a) Because of (**II**) the function f satisfies (17) with $p = 1$, i.e., $f(x) = g(0)f(x)$ for $x \in \mathbb{R}$. Since f is not identically zero, we get $g(0) = 1$ that is condition (18).

(b) Proceeding as in the proof of *Lemma 2 (b)* we obtain equation (14) with $p = 1$:

$$af(u - bv) + bf(u + av) = f(u)g(v), \qquad u, v \in \mathbb{R}. \tag{21}$$

According to Lemma 2 we may differentiate both sides of (21) with respect to v (in J) twice, to arrive at

$$ab^2 f''(u - bv) + a^2 bf''(u + av) = f(u)g''(v) \quad u \in \mathbb{R},\ v \in J.$$

With $v = 0$ ($\in J$) and $u = x$ here, thanks to (**II**) we get (20). □

7. Case $a + b = 1$; main result

In the theorem that follows exponential functions show up.

Theorem 3. *Assume* (**II**) *and let* $(f, g) \in \mathcal{F} \times \mathcal{G}$ *be a solution to* (1) *satisfying*

$$g''(0) > 0. \tag{$\star$}$$

If $f(0) \neq 0$ *then there is a non-zero* c *such that either*

$$f(x) = ce^{kx};\ g(x) = be^{akx} + ae^{-bkx}, \quad x \in \mathbb{R}, \tag{22}$$

or

$$f(x) = ce^{-kx};\ g(x) = ae^{bkx} + be^{-akx}, \quad x \in \mathbb{R}, \tag{23}$$

with

$$k := (g''(0)/ab)^{1/2}. \tag{24}$$

If $f(0) = 0$, *then* f *and* g *are given by* (8).

Proof. 1° If $(f, g) \in \mathcal{F} \times \mathcal{G}$ satisfy equation (1), then f fulfils equation (20) which, according to ($\star$) and (24), becomes $f''(x) = k^2 f(x)$ and

$$f(x) = Ae^{kx} + Be^{-kx}, \qquad x \in \mathbb{R}, \tag{25}$$

with some constants A and B.

2° Assume that $f(x) \neq 0$ in $\mathbb{R}$. Putting $x = 0$ in (1) we calculate (for $y \in \mathbb{R}$)

$$g(y) = \frac{af(0) + bf(y)}{f(by)} \tag{26}$$

and eliminate g from (1):

$$[af(x) + bf(y)]f[b(y - x)] = f(ax + by)[af(0) + bf(y - x)], \quad x, y \in \mathbb{R}.$$

We may substitute here $y = 0$. The resulting equation takes the form:

$$[af(x) + bd]f(-bx) = f(ax)[bf(-x) + ad], \quad x \in \mathbb{R} \quad (d := f(0)). \tag{27}$$

Using (25) in (27) we get the identity (with $t := kx$, for short):

$$\sum_{j=1}^{8} \alpha_j \exp(m_j t) \equiv 0, \tag{28}$$

where

$$m_1 = a, \quad m_2 = -a, \quad m_3 = b, \quad m_4 = -b,$$

$$m_5 = 1 + a, \quad m_6 = -(1 + a), \quad m_7 = 1 + b, \quad m_8 = -(1 + b)$$

and

$$\alpha_1 = aA(A - d), \qquad \alpha_2 = aB(B - d),$$

$$\alpha_3 = bB(d - B), \qquad \alpha_4 = bA(d - A),$$

$$\alpha_5 = \alpha_6 = bAB, \qquad \alpha_7 = \alpha_8 = aAB.$$

Since the exponents in (28) are mutually different, the corresponding exponential functions are linearly independent. The equalities $\alpha_5 = \cdots = \alpha_8 = 0$ yield $A = 0$ or $B = 0$. When $A = 0$, from the equalities $\alpha_2 = \alpha_3 = 0$ we get $B = d$, whereas when $B = 0$ there is $A = d$, because of $\alpha_1 = \alpha_4 = 0$. We have found formula (22), resp. (23), for f, with $C = d$.

We proceed with determining $g(x)$ from (26). At first we put $f(x) = de^{kx}$ there. We obtain by (**II**) $g(x) = be^{akx} + ae^{-bkx}$, in accordance with (22). Formula (23) for $g(x)$, corresponding to $f(x) = de^{-kx}$ is obtained in the same way.

A straightforward calculation shows that the functions f (which does not vanish on $\mathbb{R}$) and g given by (22) or (23) (with (24)) actually satisfy (1) and that they are from $\mathcal{F} \times \mathcal{G}$.

3° In turn, let $f(0) \neq 0$ and $f(r) = 0$ for an $r > 0$. We make use of (21) putting $u = r$, $v = t$ there:

$$af(r - bt) + bf(r + at) = 0, \quad -\frac{r}{a} \leq t \leq \frac{r}{b}.$$

On replacing t by $-t$ we obtain

$$af(r + bt) + bf(r - at) = 0, \quad -\frac{r}{b} \leq t \leq \frac{r}{a}.$$

The equalities when first added then subtracted side by side yield

$$a\varphi(t) + b\varphi\left(\frac{a}{b}t\right) = 0, \quad a\psi(t) - b\psi\left(\frac{a}{b}t\right) = 0, \quad t \in I_r := \left[-\frac{r}{b}, \frac{r}{b}\right], \tag{29}$$

where we have put

$$\begin{aligned} &\varphi(t) := f(r + bt) + f(r - bt), \\ &\psi(t) := f(r + bt) - f(r - bt), \quad t \in I_r. \end{aligned} \tag{30}$$

Equations (29) are the Schröder equations (4). We rewrite the first equation in (29) as

$$\varphi\left(\frac{a}{b}t\right) = -\frac{a}{b}\varphi(t), \qquad t \in I_r,$$

and replace t by $ab^{-1}t$. Hence we get

$$\varphi\left(\frac{a^2}{b^2}t\right) = -\frac{a}{b}\varphi\left(\frac{a}{b}t\right) = \frac{a^2}{b^2}\varphi(t), \qquad t \in I_r,$$

i.e., equation (4) with $0 < p = q = a^2b^{-2} < 1$ (cf. (**II**)). In turn, cf. (29),

$$\psi\left(\frac{a}{b}t\right) = \frac{a}{b}\psi(t), \qquad t \in I_r,$$

i.e., equation (4) with $0 < p = q = ab^{-1} < 1$ (cf. (**II**)). By Lemma 2 the function f is of class $C^1(\mathbb{R})$, whence so are φ and ψ given by (30), in the interval I. Lemma 1(B) then implies that there are real constants α_1 and α_2 such that $\varphi(t) = \alpha_1 t$, $\psi(t) = \alpha_2 t$, $t \in I_r$. From relations (30) we see that $2f(r+bt) = \varphi(t) + \psi(t) = (\alpha_1 + \alpha_2)t$ for $t \in I_r$. This means that, whenever $x \in [0, 2r]$, we have

$$f(x) = \gamma(x - r), \tag{31}$$

where $\gamma := \frac{1}{2}(\alpha_1 + \alpha_2)$. Coming back to equation (1) we take $x, y \in [0, 2r]$ there. Since the convex combination $ax + by$ of x and y (cf. (**II**)) is also in $[0, 2r]$, we have by (31):

$$a\beta \cdot (x - r) + b\beta \cdot (y - r) = [\beta \cdot (ax + by) - r]g(y - x).$$

As $a + b = 1$, this yields $g(y - x) = 1$ for $x, y \in [0, 2r]$, that is $g(t) = 1$ for $|t| \leq 2r$. Therefore $g''(0) = 0$, which contradicts assumption $(\star)$.

If $f(r) = 0$ for an $r < 0$, the proof runs the same way.

4° Finally, in the case where $f(0) = 0$ we get from (25) the formula

$$f(x) = C \sinh kx, \qquad x \in \mathbb{R}, \tag{32}$$

where $C := 2A = -2B$. Assume that $C \neq 0$. Thus $f(x) \neq 0$ for $x \neq 0$ and formula (26) works for $y \neq 0$, yielding

$$g(y) = \frac{bf(y)}{f(by)} = \frac{b\ \sinh(ky)}{\sinh(kby)}, \qquad y \neq 0. \tag{33}$$

On letting $y = 0$ in (1) and taking into account that $f(0) = 0$ we obtain

$$af(x) = f(ax)g(-x).$$

Substituting here (32) and (33) we have $a \sinh(kbx) = b \sinh(kax)$ for $x \neq 0$ which is not an identity when (**II**) is assumed. Therefore $C = 0$ in (32), whence f is the zero function and g is arbitrary, i.e., formula (8) describes all the solutions of (1). $\square$

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