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On the m-convexity of $C_b(X)$

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Abstract. Let X be a topological space and $C_b(X)$ the algebra of bounded continuous complex functions defined on X, with the strict topology β defined by R. Giles. In this paper a necessary and sufficient condition is given in order that $C_b(X)$ be an m-convex algebra, when X is a completely regular Hausdorff space. The density of principal ideals in this algebra and an algebra of analytic sequences are also studied.

1. Introduction

Let X be a topological space. We denote by $B\,(X)$ the algebra of all bounded complex functions on X, and by $C_b\,(X)$ the subalgebra of $B\,(X)$ consisting of bounded continuous functions. The ideal in $B\,(X)$ of all bounded functions vanishing at infinity is denoted by $B_0(X)$ and $B_{00}(X)$ denotes the subspace of $B_0(X)$ consisting of all the elements in $B(X)$ with compact support.

The strict topology β on the algebra $C_b(X)$ was introduced by C. Buck in [4] when X is a locally compact Hausdorff space. For an arbitrary topological space X it was defined by R. Giles [5] as the locally convex topol-

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ogy on $C_b(X)$ given by the seminorms

$$\|f\|_\varphi = \sup_{x\in Y} |f(x)\varphi(x)|, \tag{1}$$

where φ ranges on $B_0(X)$. If we restrict the functions φ to the class $B_{00}(X)$ we obtain the compact-open topology κ, and we obtain the uniform convergence topology σ defined by the sup norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$, if we allow φ to be any function in $B(X)$. This shows that $\kappa \preccurlyeq \beta \preccurlyeq \sigma$.

If X is a locally compact space, then the strict topology β on the algebra $C_b(X)$ can be defined by the family of seminorms (1), but with φ restricted to the space $C_0(X) = C_b(X) \cap B_0(X)$. This is the way in which C. Buck defined the strict topology.

We recall that X is called a *k-space* if it is a space in which a set is closed iff its intersection with every compact closed set is closed. If X is a locally compact or metrizable space then X is a k-space.

In [5] it is shown that the algebra $(C_b(X), \beta)$ is complete if and only if X is a k-space.

A commutative locally convex algebra A with unit e, whose topology is given by the family $\{\| \ \|_\alpha : \alpha \in \Lambda\}$ of seminorms on A, is said to be locally A-convex if for each $x \in A$ and $\alpha \in \Lambda$ there exists some constant $M(x,\alpha) > 0$ such that

$$\|xy\|_\alpha \leq M_{(x,\alpha)} \|y\|_\alpha \quad \text{for all } y \in A. \tag{2}$$

If the above constant $M_{(x,\alpha)}$ does not depend on α i.e. (2) holds for all $\alpha \in \Lambda$ and some constant M_x depending only on x, then we say that A is a locally uniformly A-convex algebra.

We say that A is a locally m-convex (shortly m-convex) algebra if every seminorm $\| \ \|_\alpha$ is submultiplicative i.e. $\|xy\|_\alpha \leq \|x\|_\alpha \|y\|_\alpha$ for all $\alpha \in \Lambda$ and $x, y \in A$.

The algebra $(C_b(X), \beta)$ is locally uniformly A-convex, since $\|fg\|_\phi \leq \|f\|_\infty \|g\|_\phi$ for every $\phi \in B_0(X)$ and $f, g \in C_b(X)$. It is easy to see that the topological algebras $(C_b(X), \sigma)$ and $(C_b(X), \kappa)$ are m-convex algebras. In this paper we establish, among other things, some conditions under which the algebra $(C_b(X), \beta)$ is also an m-convex algebra.

By $\mathcal{M}\,(A)$ (resp., $\mathcal{M}^{\#}(A)$) we denote the space of all continuous nonzero linear multiplicative complex functionals on A (resp., all non zero linear multiplicative complex functionals on A).

If X is a completely regular Hausdorff space, then it is well known that $\mathcal{M}\left(C_b\left(X\right),\beta\right) = X$, i.e. $h \in \mathcal{M}\left(C_b(X),\beta\right)$ if and only if $h(f) = \widehat{x}(f)$ for all $f \in C_b(X)$ and a fixed $x \in X$, where $\widehat{x}\,(f) = f(x)$.

2. The Wiener property

A commutative complete complex m-convex algebra A with unit satisfies the *Wiener property: $x \in A$ is invertible if and only if $\widehat{x}(f) \neq 0$ for every $f \in \mathcal{M}(A)$.*

In this section we formulate for $(C_b(X),\beta)$ a result, Corollary 2.2, that resembles the Wiener property and we use this result to prove that a particular commutative locally convex complete algebra with unit is not m-convex.

The next theorem is the complex version of the Stone–Weierstrass theorem given in [5].

Theorem 2.1. *Let A be a self adjoint β-closed subalgebra of $C_b(X)$ which separates points and contains, for each x in X, a function nonvanishing at x. Then $A = C_b\,(X)$.*

Corollary 2.2. *Let X be a completely regular Hausdorff space. Suppose $f \in C_b(X)$ is such that $f(x) \neq 0$ for every $x \in X$. Then the ideal $fC_b\,(X)$ is dense in $(C_b(X),\beta)$.*

Proof. Since X is a completely regular Hausdorff space, $C_b(X)$ separates points and so does $fC_b(X)$, and since $\frac{\overline{f}}{f}\,\overline{g} \in C_b(X)$ for every $g \in C_b\,(X)$, $fC_b(X)$ is self adjoint. $\square$

When the above function f is not invertible in $C_b(X)$ we obtain the following

Theorem 2.3. *If $f \in C_b(X)$ is such that $f(x) \neq 0$ for every $x \in X$ and $\inf_{x\in X} |f(x)| = 0$, then the ideal $fC_b(X)$ is of infinite codimension.*

Proof. Let us assume that f satisfies the hypothesis. For each $n \geq 1$ let us define the function $h_n(x) = \sqrt[2^n]{|f(x)|}$ for all $x \in X$. Then we obtain a sequence $(h_n C_b(X))_{n=1}^{\infty}$ of ideals of $C_b(X)$ such that

$$h_1 C_b(X) \subsetneq h_2 C_b(X) \subsetneq \dots .$$

This implies that $\{h_n : n \geq 1\}$ is a set of linearly independent elements. Since the function $\frac{g}{f}$ is not bounded whenever g is not the null element in the linear space $\langle h_n \rangle$ generated by the set $\{h_n : n \geq 1\}$, it follows that $\langle h_n \rangle \cap f C_b(X) = \{0\}$ and then the ideal $f C_b(X)$ is of infinite codimension. □

Corollary 2.4. *Let X be a completely regular Hausdorff k-space. If there exists $f \in C_b(X)$ as in the above theorem, then $(C_b(X), \beta)$ is not an m-convex algebra.*

Proof. We know that $\mathcal{M}(C_b(X), \beta) = X$, and by hypothesis $\widehat{x}(f) = f(x) \neq 0$ for every $x \in X$. Therefore, $(C_b(X), \beta)$ is a commutative complete complex algebra with unit that does not satisfy the Wiener condition. Thus, $(C_b(X), \beta)$ is not an m-convex algebra. □

3. The $\mathcal{M}$-convexity of $(C_b(X), \beta)$

Let A be a topological algebra. In [2] an element $x \in A$ is said to be $\mathcal{M}$-invertible (resp., $\mathcal{M}^{\#}$-invertible) if $\widehat{x}(f) \neq 0$ for every $f \in \mathcal{M}(A)$ (resp., $f \in \mathcal{M}^{\#}(A)$). The set of all $\mathcal{M}$-invertible ($\mathcal{M}^{\#}$-invertible) elements in A is denoted by $G_{\mathcal{M}}(A)$ ($G_{\mathcal{M}^{\#}}(A)$). The set of all invertible elements in A is denoted, as usual, by $G(A)$.

Suppose X is a completely regular Hausdorff space. Since $\mathcal{M}(C_b(X), \beta) = X$ and $\mathcal{M}^{\#}(C_b(X)) = \beta(X)$ (the Stone–Čech compactification of X) we have

$$G_{\mathcal{M}}(C_b(X), \beta) = \{f \in C_b(X) : f(x) \neq 0, \ \forall\, x \in X\}$$

and

$$G(C_b(X)) = G_{\mathcal{M}^{\#}}(C_b(X)) = \big\{ f \in C_b(X) : \inf_{x \in X} |f(x)| > 0 \big\}.$$

Proposition 3.1. *Suppose X is a completely regular Hausdorff space. The following properties are equivalent:*

(1) $C_b(X) = C(X)$, *where $C(X)$ is the space of all complex continuous functions on X.*

(2) $G\left(C_b(X)\right) = G_{\mathcal{M}}\left(C_b(X), \beta\right)$.

Proof. It is obvious that (1) $\Rightarrow$ (2). To show (2) $\Rightarrow$ (1) we assume the contrary, namely that there exists $f \in C(X)$ which is not a bounded function. Then $1+|f(x)| \neq 0$ for every $x \in X$ and it is not a bounded function. Therefore, the element $h = \frac{1}{1+f}$ belongs to $G_{\mathcal{M}}(C_b(X), \beta)\diagdown G\left(C_b\left(X\right)\right)$. $\square$

A topological algebra A is said to be a Q-algebra if $G\left(A\right)$ is an open set in A, in other words the complement of $G\left(A\right)$ is closed in A. In the topological algebra $(C_b(X), \beta)$, with X a completely regular non compact Hausdorff space, the set of invertible elements has the opposite property, as we can see in the following

Proposition 3.2. *Let X be a completely regular noncompact Hausdorff space. The set of all noninvertible elements of $C_b\left(X\right)$ is dense in $\left(C_b\left(X\right), \beta\right)$.*

Proof. Let $\varphi \in B_0\left(X\right)$ and $\epsilon > 0$. There exists a compact subset $K \subset X$ such that $|\varphi(x)| < \epsilon$ for every $x \notin K$. Since X is a completely regular noncompact Hausdorff space there exist $x_0 \notin K$ and a function $g \in C_b(X)$ such that $g(x) = 1$ if $x \in K$, $g\left(x_0\right) = 0$ and $0 \leq g(x) \leq 1$ for all $x \in X$. It immediately follows that g is not invertible in $C_b\left(X\right)$ and $\left\|g - 1\right\|_{\varphi} < \epsilon$. $\square$

In what follows we establish a necessary and sufficient condition for the m-convexity of $(C_b(X), \beta)$, when X is a completely regular Hausdorff space. For this we follow the proof of Proposition 4 in [11].

Theorem 3.3. *Let X be a completely regular Hausdorff space. $(C_b(X), \beta)$ is an m-convex algebra if and only if $B_0\left(X\right) = B_{00}\left(X\right)$.*

Proof. Let us assume that $B_{00}(X) \subsetneq B_0(X)$ and suppose that $(C_b(X), \beta)$ is an m-convex algebra; so there exists a system P of submultiplicative seminorms that defines β. Thus, for $\varphi \in B_0(X)\diagdown B_{00}(X)$ we can find a submultiplicative seminorm $\| \ \|$ belonging to P and two positive constants p and q such that

$$p \|f\|_\varphi \leq \|f\| \leq q \|f\|_\varphi$$

for all $f \in C_b(X)$. Then $\|f\| < 1$ whenever $q \|f\|_\varphi < 1$, and so $\|f^n\| < 1$ and $p \|f^n\|_\varphi < 1$ for all $n \geq 1$. Since $\lim_{x \to \infty} \varphi(x) = 0$ we can find a compact subset K of X such that $q |\varphi(x)| < \frac{1}{2}$ for every $x \notin K$.

Let $f \in C_b(X)$ with $f(x) = 0$ if $x \in K$, $0 \leq f(x) \leq 2$ for all $x \in X$ and $f(x_1) = 2$ for some $x_1 \notin K$ for which $\varphi(x_1) \neq 0$. We have that $q \|f\|_\varphi < 1$, and then $p \|f^n\|_\varphi < 1$ for all $n \geq 1$. On the other hand,

$$p \|f^n\|_\varphi \geq 2^n |\varphi(x_1)| p$$

for all $n \geq 1$ and the expression on the right tends to ∞ as n grows. This shows that $(C_b(X), \beta)$ is a non m-convex algebra.

If we have $B_0(X) = B_{00}(X)$ then the topology β coincides with κ and then

$$(C_b(X), \beta) = (C_b(X), \kappa)$$

is clearly m-convex. □

Corollary 3.4. *Let X be a locally compact Hausdorff space. $(C_b(X), \beta)$ is an m-convex algebra if and only if $C_0(X) = C_{00}(X)$.*

Proof. If $(C_b(X), \beta)$ is an m-convex algebra, then $B_0(X) = B_{00}(X)$. If $f \in C_0(X)$, then $f \in B_{00}(X)$. Thus, $f \in C_{00}(X)$.

Conversely, since X is a locally compact Hausdorff space, the strict and the uniform topologies in C_b are given by the families of seminorms $\{\| \ \|_\varphi : \varphi \in C_0(X)\}$ and $\{\| \ \|_\varphi : \varphi \in C_{00}(X)\}$, respectively. Thus, these two topologies coincide and $(C_b(X), \beta)$ is an m-convex algebra. □

Remark 3.5. Observe that $C_b(X) = C(X)$, where X is a locally compact space, does not imply in general that $C_0(X) = C_{00}(X)$, as we can see in the following example:

Let Ω and ω be the first uncountable and countable ordinal numbers, respectively. It can be proved that the space

$$Y = [0, \Omega] \times [0, \omega] - (\Omega, \omega)$$

is pseudocompact and so $C_b(Y) = C(Y)$. Let $f : Y \to \mathbb{C}$ be defined as $f(\alpha, 0) = 1$, $f(\alpha, \omega) = 0$ and $f(\alpha, n) = \frac{1}{n}$ for every $\alpha \in [0, \Omega]$ and $n = 1, 2, \ldots$. It is easy to see that $f \in C_0(Y) \diagdown C_{00}(Y)$. Thus, $(C_b(X), \beta)$ is not m-convex.

A space X is called a P-*space* if every function in $C_0(X)$ is constant in some neighborhood of each point of X. If X is a locally compact Hausdorff space such that its Stone–Ĉech compactification is a P-space, then $C_0(X)$ coincides with $C_{00}(X)$ and therefore $(C_b(X), \beta)$ is an m-convex algebra. For example, every ordinal segment $[0, \tau)$, where τ is an infinite ordinal with uncountable cofinality, has this property.

In [1], for a locally A-convex algebra $(A, \tau(P))$ with unit, where $P = \{p_\alpha \mid \alpha \in \Lambda\}$ is a family of seminorms which determines the topology τ, another topology $\tau(\widetilde{P})$ is defined. This topology $\tau(\widetilde{P})$ is the weakest locally m-convex topology on A which is stronger than $\tau(P)$, and it is given by the family of seminorms $\widetilde{P} = \{\widetilde{p}_\alpha \mid \alpha \in \Lambda\}$, where

$$\widetilde{p}_\alpha(x) = \sup\{p_\alpha(xy) : p_\alpha(y) \leq 1\}.$$

For $(C_b(X), \beta)$ this locally m-convex topology will be denoted by $\beta(\widetilde{P})$ and it is defined by the seminorms

$$\widetilde{p}_\varphi(f) = \sup\left\{\|fg\|_\varphi : \|g\|_\varphi \leq 1\right\},$$

where $f, g \in C_b(X)$ and $\varphi \in B_0(X)$.

The following lemma is obvious.

Lemma 3.6. *Let X be a locally compact Hausdorff space. There exists a real function $\varphi \in B_0(X)$ such that $\varphi(x) \neq 0$ for all $x \in X$ if and only if X is σ-compact.*

Proposition 3.7. *If X is a locally compact and σ-compact Hausdorff space then the topology $\beta(\widetilde{P})$ in $C_b(X)$ coincides with the uniform topology.*

Proof. It is clear that

$$\widetilde{p}_\varphi(f) \leq \|f\|_\infty$$

for all $f \in C_b(X)$ and $\varphi \in B_0(X)$.

On the other hand, by the above lemma there exists $\varphi \in B_0(X)$ such that $\varphi(x) \neq 0$ for all $x \in X$. Given $\epsilon > 0$ and $f \in C_b(X)$, let $x \in X$ be such that

$$\|f\|_\infty - \epsilon < |f(x)|,$$

then

$$\|f\|_\infty - \epsilon < |f(x)| = \left| f(x) \frac{1}{\varphi(x)} \varphi(x) \right| \leq \|f g_\epsilon\|_\varphi,$$

where $g_\epsilon(x) = \frac{1}{\varphi(x)}$ and $g_\epsilon(y) = 0$ if $y \neq x$. Thus, $\|f\|_\infty \leq \widetilde{p}_\varphi(f)$.

The space $[0, \Omega)$ is a locally compact Hausdorff space, but it is not a σ-compact space. In this case, the $\beta(\widetilde{P})$ topology coincides with the open-compact topology in $C_b([0, \Omega))$.

On the other hand, $\beta(\widetilde{P})$ coincides with the uniform topology in the space Y of Remark 3.5, because $\|f\|_\infty = \widetilde{p}(f)$ for the function f defined there. □

4. The algebra $H(D)$

Let $H(D)$ be the algebra of all holomorphic functions in the unit complex open disc D, and let A denotes the space of all complex sequences $\mathfrak{a} = (a_k)_{k=0}^\infty$ such that if z is a complex number and $|z| < 1$, then $\sum_{k=0}^\infty a_k z^k$ converges. The transformation

$$f(z) = \sum_{k=0}^{\infty} a_k(f) z^k \to \mathfrak{a}(f) = (a_k(f))_{k=0}^\infty \tag{3}$$

identifies $H(D)$ with the sequence space A.

Let A be endowed with the Hadamard product, i.e. the coordinatewise product, and the compact-open topology inherited from $H(D)$ through the identification (3); this topology, that we denote by $\tau(A)$, can be given by the sequence $(\| \ \|_n)_{n=1}^\infty$ of seminorms on A defined as

$$\|(a_k(f))_{k=0}^\infty\|_n = \sup_{k \geq 0} \left(|a_k(f)| r_n^k \right),$$

for $n \geq 1$, where $(r_n)_{n=1}^\infty$ is an increasing sequence of positive numbers tending to 1.

Then A becomes an algebra of analytic sequences, moreover, $(A, \tau(A))$ is a locally convex, metrizable complete commutative algebra with unit $e = (1, 1, \dots)$ and orthogonal basis $(e_n)_{n=0}^{\infty}$, where $e_{nk} = \delta_{nk}$ for $n, k \geq 0$.

In [3], it is proved that $(a_k(f))_{k=0}^{\infty} \in A$ is invertible if and only if it satisfies

i) $a_k(f) \neq 0$ for every $k \geq 0$ and

ii) $\lim_{k\to\infty} |a_n(f)|^{1/k} = 1$.

Now we prove the following

Proposition 4.1. *If $\mathfrak{a}(f) = (a_k(f))_{k=0}^{\infty}$ in A is such that $a_k(f) \neq 0$ for every $k \geq 0$, then $\mathfrak{a}(f)A$ is dense in $(A, \tau(A))$ and if $\mathfrak{a}(f)$ is not invertible, then the ideal $\mathfrak{a}(f)A$ is of infinite codimension.*

Proof. Let us assume first that $\mathfrak{a}(f) \in \ell^{\infty}$, then by Theorem 2.2 we have that $\mathfrak{a}(f)l^{\infty}$ is dense in (ℓ^{∞}, c_0) and so, for each $\mathfrak{j} \in l^{\infty}$, $\mathfrak{b} \in c_0$ and $\epsilon > 0$ there exists $\mathfrak{h} \in \ell^{\infty}$ such that

$$\sup_{k\geq 0} |(j_k - a_k(f)h_k)b_k| < \epsilon.$$

In particular, for the sequence $(r_n)_{n=1}^{\infty}$ we have $\left(r_n^k\right)_{k=0}^{\infty} \in c_0$ for each positive integer $n \geq 1$, and so

$$\sup_{k\geq 0} \left|(j_k - a_k(f)h_k)r_n^k\right| < \epsilon.$$

This implies that $\ell^{\infty} \subset \overline{\mathfrak{a}(f)A}$ (the $\tau(A)$-closure of $\mathfrak{a}(f)A$) and since ℓ^{∞} is dense in $(A, \tau(A))$, it follows that $\mathfrak{a}(f)A$ is dense in A with the compact-open topology $\tau(A)$.

If $\mathfrak{a}(f) \in A$ is such that $\mathfrak{a}(f) \notin \ell^{\infty}$, then there exists $\mathfrak{b} \in A$ such that $\mathfrak{a}(f)\mathfrak{b} \in \ell^{\infty}$ and so we are led to the previous case.

On the other hand, if $\mathfrak{a}(f) \in A$ is such that $a_k(f) \neq 0$ for all $k = 0, 1, \dots$, and it is not invertible, then $\mathfrak{a}(f) \notin M_k$, where $M_k = \{\mathfrak{a}(g) \in A : a_k(g) = 0\}$ for each $k \geq 0$, and therefore $\mathfrak{a}(f)A$ cannot be contained in any M_k, but each proper ideal is contained in some maximal ideal; so $\mathfrak{a}(f)A$ must be contained in some ideal M^p with $p \in \beta(\mathbb{N}) \setminus \mathbb{N}$.

Since the algebra A is functionally continuous (see [3]), this ideal M^p is dense of infinite codimension and hence $\mathfrak{a}(f)A$ is of infinite codimension.

□

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