

Existence of solutions for nonlinear second order systems on a measure chain

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Abstract. Under suitable conditions on positive functions $f(t, v)$ and $g(t, u)$, we prove that the nonlinear second order systems on a measure chain

$$(BVPS) \quad \left\{ \begin{array}{ll} (E_1) & u^{\Delta\Delta}(t) + f(t, v(\sigma(t))) = 0, \quad 0 < t < 1, \\ (E_2) & v^{\Delta\Delta}(t) + g(t, u(\sigma(t))) = 0, \quad 0 < t < 1, \\ (BC_1) & \left\{ \begin{array}{l} \alpha_1 u(0) - \beta_1 u^\Delta(0) = 0, \\ \gamma_1 u(\sigma(1)) + \delta_1 u^\Delta(\sigma(1)) = 0, \end{array} \right. \\ (BC_2) & \left\{ \begin{array}{l} \alpha_2 v(0) - \beta_2 v^\Delta(0) = 0, \\ \gamma_2 v(\sigma(1)) + \delta_2 v^\Delta(\sigma(1)) = 0, \end{array} \right. \end{array} \right.$$

has at least one positive solution.

1. Introduction

In 1990, S. HILGER [8] introduced the theory of measure chain in order to unify continuous and discrete calculus. Recently, the development of theory of measure chain has received a lot of attention, see [1]–[7], [9], [11], [12].

Mathematics Subject Classification: 34B15.

Key words and phrases: measure chain, boundary value problem systems, positive solution, cone, fixed point.

The existence of solutions of the boundary value problem

$$(BVP) \quad \begin{cases} (E) & u^{\Delta\Delta}(t) + f(t, u(\sigma(t))) = 0, \quad 0 < t < 1, \\ (BC) & \begin{cases} \alpha u(0) - \beta u^{\Delta}(0) = 0, \\ \gamma u(\sigma(1)) + \delta u^{\Delta}(\sigma(1)) = 0 \end{cases} \end{cases}$$

on a measure chain has been studied by many authors, see, for examples, CHYAN and HENDERSON [4], ERBE and PETERSON [7], HONG and YEH [9] and W. C. LIAN, C. C. CHOU, C. T. LIU and F. H. WONG [12].

In this article, we shall consider the existence of positive solutions of the following boundary value problem systems

$$(BVPS) \quad \begin{cases} (E_1) & u^{\Delta\Delta}(t) + f(t, v(\sigma(t))) = 0, \quad 0 < t < 1, \\ (E_2) & v^{\Delta\Delta}(t) + g(t, u(\sigma(t))) = 0, \quad 0 < t < 1, \\ (BC_1) & \begin{cases} \alpha_1 u(0) - \beta_1 u^{\Delta}(0) = 0, \\ \gamma_1 u(\sigma(1)) + \delta_1 u^{\Delta}(\sigma(1)) = 0, \end{cases} \\ (BC_2) & \begin{cases} \alpha_2 v(0) - \beta_2 v^{\Delta}(0) = 0, \\ \gamma_2 v(\sigma(1)) + \delta_2 v^{\Delta}(\sigma(1)) = 0, \end{cases} \end{cases}$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i$ are nonnegative real numbers and $r_i := \gamma_i \beta_i + \alpha_i \delta_i + \alpha_i \gamma_i \sigma(1) > 0$, $i = 1, 2$ and $f, g \in C_{rd}([0, \sigma(1)] \times [0, \infty), (0, \infty))$.

2. Main results

In order to abbreviate our discussion, throughout this paper, we suppose that the following assumptions hold:

(C₁) $\xi := \min \left\{ t \in T \mid t \geq \frac{\sigma(1)}{4} \right\}$ and $\omega := \max \left\{ t \in T \mid t \leq \frac{3\sigma(1)}{4} \right\}$ both exist and satisfy

$$\frac{\sigma(1)}{4} \leq \xi < \omega \leq \frac{3\sigma(1)}{4}.$$

(C₂) $G_i(t, s)$ is the Green's function of the differential equation

$$-u^{\Delta\Delta}(t) = 0 \quad \text{in } (0, 1)$$

satisfying the boundary value condition (BC_i);

(C₃) $M_i = \min\{d_i, l_i\}$, where

$$d_i := \min \left\{ \frac{\gamma_i \sigma(1) + 4\delta_i}{4(\gamma_i \sigma(1) + \delta_i)}, \frac{\alpha_i \sigma(1) + 4\beta_i}{4(\alpha_i \sigma(1) + \beta_i)} \right\} \in (0, 1)$$

and

$$l_i = \min_{s \in [0, \sigma(1)]} \frac{G_i(\sigma(\omega), s)}{G_i(\sigma(s), s)}.$$

(C₄) $f, g \in C([0, \sigma(1)] \times [0, \infty); (0, \infty))$.

In order to prove our main result (Theorem 2.1 below), we shall need the following two useful lemmas:

Lemma 2A (ERBE and PETERSON [7]). *Let M_i be defined as in (C₃). For the Green's function $G_i(t, s)$ ($i = 1, 2$) the following results hold:*

$$\begin{cases} \text{(R}_1\text{)} & \frac{G_i(t, s)}{G_i(\sigma(s), s)} \leq 1 \quad \text{for } t \in [0, \sigma(1)] \text{ and } s \in [0, \sigma(1)], \\ \text{(R}_2\text{)} & \frac{G_i(t, s)}{G_i(\sigma(s), s)} \geq M_i \quad \text{for } t \in [\xi, \omega] \text{ and } s \in [0, \sigma(1)]. \end{cases}$$

Lemma 2B (KRASNOSELSKII [10]). *Let $P \subseteq E$ be a cone in a Banach space E . Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$. If*

$$\Phi : P \cap (\overline{\Omega_2} \setminus \Omega_1) \longrightarrow P$$

is a completely continuous operator such that either

- (i) $\|\Phi u\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|\Phi u\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$; or
- (ii) $\|\Phi u\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|\Phi u\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$,

then Φ has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Denote the Banach space $E = \{\mathcal{U} = (u, v) \in (C[0, \sigma(1)])^2\}$ with norm

$$\|\mathcal{U}\| = \|(u, v)\| := \max\{\|u\|_\infty, \|v\|_\infty\}.$$

Here $\|u\|_\infty := \sup_{0 \leq t \leq \sigma(1)} |u(t)|$. Define a set $P \subset E$ by

$$P := \left\{ (u, v) \mid (u, v) \geq (0, 0), \right. \\ \left. \left(\min_{\xi \leq t \leq \sigma(\omega)} u(t), \min_{\xi \leq t \leq \sigma(\omega)} v(t) \right) \geq (M_1 \|u\|_\infty, M_2 \|v\|_\infty) \right\},$$

where $(a, b) \geq (c, d)$ means that $a \geq c$ and $b \geq d$. Here $a, b, c, d \in \mathfrak{R}$. It is clear that P is a cone in E .

Now, we can state and prove our main result.

Theorem 2.1 (Main result). *Assume that there exist four positive constants $\eta_1, \eta_2, \lambda_1$ and λ_2 such that for $(v, u) \in [0, \lambda_1] \times [0, \lambda_2]$,*

$$\left(\int_0^{\sigma(1)} G_1(t, s) f(s, v) \Delta s, \int_0^{\sigma(1)} G_2(t, s) g(s, u) \Delta s \right) \leq (\lambda_1, \lambda_2) \quad (1)$$

and for $(v, u) \in [M_1 \eta_1, \eta_1] \times [M_2 \eta_2, \eta_2]$,

$$\left(\int_{\xi}^{\omega} G_1(\theta, s) f(s, v) \Delta s, \int_{\xi}^{\omega} G_2(\theta, s) g(s, u) \Delta s \right) \geq (\eta_1, \eta_2), \quad (2)$$

where $\theta \in (\xi, \omega)$. Then (1) has at least one positive solution (u, v) between λ and η , where $\lambda := \max\{\lambda_1, \lambda_2\}$ and $\eta := \min\{\eta_1, \eta_2\}$.

PROOF. Without loss of generality, we may assume that $\lambda < \eta$. It is clear that (1) has a solution $\mathcal{U} := (u, v) = (u(t), v(t))$ if and only if $\mathcal{U}$ is the solution of the operator equation

$$\begin{aligned} \Phi \mathcal{U}(t) &:= (\Phi u(t), \Phi v(t)) \\ &:= \left(\int_0^{\sigma(1)} G_1(t, s) f(s, v(\sigma(s))) \Delta s, \int_0^{\sigma(1)} G_2(t, s) g(s, u(\sigma(s))) \Delta s \right) \\ &= \mathcal{U}(t) \text{ for } t \in [0, \sigma(1)] \text{ and } \mathcal{U} \in E. \end{aligned}$$

It follows from the definition of P and Lemma 2A that

$$\begin{aligned} &\min_{t \in [\xi, \omega]} (\Phi \mathcal{U})(t) \\ &= \left(\min_{t \in [\xi, \omega]} \int_0^{\sigma(1)} G_1(t, s) f(s, v(\sigma(s))) \Delta s, \min_{t \in [\xi, \omega]} \int_0^{\sigma(1)} G_2(t, s) g(s, u(\sigma(s))) \Delta s \right) \\ &\geq \left(M_1 \int_0^{\sigma(1)} G_1(\sigma(s), s) f(s, v(\sigma(s))) \Delta s, M_2 \int_0^{\sigma(1)} G_2(\sigma(s), s) g(s, u(\sigma(s))) \Delta s \right) \\ &\geq \left(M_1 \int_0^{\sigma(1)} G_1(t, s) f(s, v(\sigma(s))) \Delta s, M_2 \int_0^{\sigma(1)} G_2(t, s) g(s, u(\sigma(s))) \Delta s \right) \end{aligned}$$

and

$$\begin{aligned}
& (\Phi \mathcal{U})(\sigma(\omega)) \\
&= \left(\int_0^{\sigma(1)} G_1(\sigma(\omega), s) f(s, v(\sigma(s))) \Delta s, \int_0^{\sigma(1)} G_2(\sigma(\omega), s) g(s, u(\sigma(s))) \Delta s \right) \\
&\geq \left(l_1 \int_0^{\sigma(1)} G_1(\sigma(s), s) f(s, v(\sigma(s))) \Delta s, l_2 \int_0^{\sigma(1)} G_2(\sigma(s), s) g(s, u(\sigma(s))) \Delta s \right) \\
&\geq \left(M_1 \int_0^{\sigma(1)} G_1(\sigma(s), s) f(s, v(\sigma(s))) \Delta s, M_2 \int_0^{\sigma(1)} G_2(\sigma(s), s) g(s, u(\sigma(s))) \Delta s \right) \\
&\geq \left(M_1 \int_0^{\sigma(1)} G_1(t, s) f(s, v(\sigma(s))) \Delta s, M_2 \int_0^{\sigma(1)} G_2(t, s) g(s, u(\sigma(s))) \Delta s \right).
\end{aligned}$$

Hence

$$\min_{t \in [\xi, \sigma(\omega)]} \Phi \mathcal{U}(t) \geq (M_1 \|\Phi u\|_\infty, M_2 \|\Phi v\|_\infty),$$

which implies $\Phi P \subset P$. Furthermore, it is easy to check that $\Phi : P \rightarrow P$ is completely continuous. In order to complete the proof, we separate the rest of the proof into the following two steps:

Step (I) Let $\Omega_1 := \{\mathcal{U} \in P \mid \|\mathcal{U}\| < \lambda\}$. It follows from (1), Lemmas 2A–2B and the fact $\mathcal{U} \in P$ that for $\mathcal{U} \in \partial\Omega_1$,

$$\begin{aligned}
\Phi \mathcal{U}(t) &= \left(\int_0^{\sigma(1)} G_1(t, s) f(s, v(\sigma(s))) \Delta s, \int_0^{\sigma(1)} G_2(t, s) g(s, u(\sigma(s))) \Delta s \right) \\
&\leq \left(\int_0^{\sigma(1)} G_1(\sigma(s), s) f(s, v(\sigma(s))) \Delta s, \int_0^{\sigma(1)} G_2(\sigma(s), s) g(s, u(\sigma(s))) \Delta s \right) \\
&\leq (\lambda_1, \lambda_2) \leq (\lambda, \lambda) = (\|\mathcal{U}\|, \|\mathcal{U}\|).
\end{aligned}$$

Hence,

$$\|\Phi \mathcal{U}\| \leq \|\mathcal{U}\| \quad \text{for } \mathcal{U} \in \partial\Omega_1.$$

Step (II) Let $\Omega_2 := \{\mathcal{U} \in P \mid \|\mathcal{U}\| < \eta\}$. It follows from the definitions of $\|\mathcal{U}\|$, P and Lemma 2B that for $\mathcal{U} \in \partial\Omega_2$,

$$\mathcal{U} = (u(t), v(t)) \leq (\|\mathcal{U}\|, \|\mathcal{U}\|) = (\eta, \eta) \text{ for } t \in [0, \sigma(1)],$$

and for $t \in [\xi, \sigma(\omega)]$,

$$\begin{aligned} (u(t), v(t)) &\geq \left(\min_{t \in [\xi, \sigma(\omega)]} u(t), \min_{t \in [\xi, \sigma(\omega)]} v(t) \right) \\ &\geq (M_1 \|u\|_\infty, M_2 \|v\|_\infty) \\ &= (M_1 \eta_1, M_2 \eta_2) \\ &\geq (M_1 \eta, M_2 \eta), \end{aligned}$$

which implies

$$(M_1 \eta, M_2 \eta) \leq (u(t), v(t)) \leq (\eta, \eta).$$

Hence, by (2),

$$\begin{aligned} (\Phi\mathcal{U})(\theta) &= \left(\int_0^{\sigma(1)} G_1(\theta, s) f(s, v(\sigma(s))) \Delta s, \int_0^{\sigma(1)} G_2(\theta, s) g(s, u(\sigma(s))) \Delta s \right) \\ &\geq \left(\int_\xi^{\omega} G_1(\theta, s) f(s, v(\sigma(s))) \Delta s, \int_\xi^{\omega} G_2(\theta, s) g(s, u(\sigma(s))) \Delta s \right) \\ &\geq (\eta_1, \eta_2) \\ &\geq (\eta, \eta) = (\|\mathcal{U}\|, \|\mathcal{U}\|). \end{aligned}$$

Thus,

$$\|\Phi\mathcal{U}\| \geq \|\mathcal{U}\| \quad \text{for } \mathcal{U} \in \partial\Omega_2.$$

Hence, by the first part of Lemma 2B, we complete the proof. $\square$

Let

$$\begin{aligned} \max h_0 &:= \lim_{w \rightarrow 0^+} \max_{t \in [0, \sigma(1)]} \frac{h(t, w)}{w}, \\ \min h_0 &:= \lim_{w \rightarrow 0^+} \min_{t \in [\xi, \sigma(\omega)]} \frac{h(t, w)}{w}, \end{aligned}$$

$$\max h_\infty := \lim_{w \rightarrow \infty} \max_{t \in [0, \sigma(1)]} \frac{h(t, w)}{w},$$

$$\min h_\infty := \lim_{w \rightarrow \infty} \min_{t \in [\xi, \sigma(w)]} \frac{h(t, w)}{w}.$$

Then we have the following:

Remark 2.2. Let α, β, γ and δ be nonnegative constants, $r := \gamma\beta + \alpha\delta + \alpha\gamma\sigma(1) > 0$,

$$M := \min \left\{ \frac{\gamma\sigma(1) + 4\delta}{4(\gamma\sigma(1) + \delta)}, \frac{\alpha\sigma(1) + 4\beta}{4(\alpha\sigma(1) + \beta)} \right\},$$

and $G(t, s)$ the Green's function of the differential equation

$$u^{\Delta\Delta}(t) = 0 \quad \text{on} \quad (0, 1)$$

satisfying the boundary value conditions

$$\begin{cases} \alpha w(0) - \beta w^\Delta(0) = 0, \\ \gamma w(\sigma(1)) + w^\Delta(\sigma(1)) = 0. \end{cases}$$

Let

$$\left(\int_0^{\sigma(1)} G(\sigma(s), s) \Delta s \right)^{-1} := A \quad \text{and} \quad \left(\int_\xi^\omega G(\theta, s) \Delta s \right)^{-1} := B.$$

$$\left(\int_0^{\sigma(1)} G_i(\sigma(s), s) \Delta s \right)^{-1} := A_i \quad \text{and} \quad \left(\int_\xi^\omega G_i(\theta, s) \Delta s \right)^{-1} := B_i, \quad (i = 1, 2).$$

Then, we have the following results.

(I) Suppose that $\max h_0 := C_1 \in [0, A)$. Taking $\epsilon = A - C_1 > 0$, there exists $\lambda_1 > 0$ (λ_1 can be chosen arbitrarily small) such that

$$\max_{t \in [0, \sigma(1)]} \frac{h(t, w)}{w} \leq \epsilon + C_1 = A \quad \text{on} \quad [0, \lambda_1].$$

Hence,

$$h(t, w) \leq Aw \leq A\lambda_1 \quad \text{on} \quad [0, \sigma(1)] \times [0, \lambda_1].$$

If we replace h by f and g , and replace A by A_1 and A_2 , respectively. Then, the hypothesis (1) of Theorem 2.1 is satisfied.

- (II) Suppose that $\min h_\infty := C_2 \in (\frac{B}{M}, \infty]$. Taking $\epsilon = C_2 - \frac{B}{M} > 0$, there exists $\eta_1 > 0$ (η_1 can be chosen arbitrarily large) such that

$$\min_{t \in [\xi, \sigma(\omega)]} \frac{h(t, w)}{w} \geq -\epsilon + C_2 = \frac{B}{M} \quad \text{on } [M\eta_1, \infty).$$

Hence,

$$h(t, w) \geq \frac{B}{M}w \geq \frac{B}{M}M\eta_1 = B\eta_1$$

on $[\xi, \sigma(\omega)] \times [M\eta_1, \eta_1]$. If we replace h by f and g , and replace B by B_1 and B_2 , respectively. Then, the hypothesis (2) of Theorem 2.1 is satisfied.

- (III) Suppose that $\min h_0 := C_3 \in (\frac{B}{M}, \infty]$. Taking $\epsilon = C_3 - \frac{B}{M} > 0$, there exists $\eta_2 > 0$ (η_2 can be chosen arbitrarily small) such that

$$\min_{t \in [\xi, \sigma(\omega)]} \frac{h(t, w)}{w} \geq -\epsilon + C_3 = \frac{B}{M} \quad \text{on } [0, \eta_2].$$

Hence,

$$h(t, w) \geq \frac{B}{M}w \geq \frac{B}{M}M\eta_2 = B\eta_2$$

on $[\xi, \sigma(\omega)] \times [0, \eta_2]$. If we replace h by f and g , and replace B by B_1 and B_2 , respectively. Then, the hypothesis (2) of Theorem 2.1 is satisfied.

- (IV) Suppose that $\max h_\infty := C_4 \in [0, A)$. Taking $\epsilon = A - C_4 > 0$, there exist $\theta > 0$ (θ can be chosen arbitrarily large) such that

$$\max_{t \in [0, \sigma(1)]} \frac{h(t, w)}{w} \leq \epsilon + C_4 = A \quad \text{on } [\theta, \infty). \quad (3)$$

Hence, we have the following two cases:

Case (i): Assume that $\max_{t \in [0, \sigma(1)]} h(t, w)$ is bounded, say,

$$h(t, w) \leq L \quad \text{on } [0, \sigma(1)] \times [0, \infty).$$

Taking $\lambda_2 = \frac{L}{A}$ (since L can be chosen arbitrarily large, λ_2 can be chosen arbitrarily large, too),

$$h(t, w) \leq L = A\lambda_2 \quad \text{on } [0, \sigma(1)] \times [0, \lambda_1] \subseteq [0, \sigma(1)] \times [0, \infty).$$

Case (ii): Assume that $\max_{t \in [0, \sigma(1)]} h(t, w)$ is unbounded, hence, there exists a $\lambda_2 \geq \theta$ (λ_2 can be chosen arbitrarily large) and $t_0 \in [0, \sigma(1)]$ such that

$$h(t, w) \leq h(t_0, \lambda_2) \quad \text{on } [0, \sigma(1)] \times [0, \lambda_2].$$

It follows from $\lambda_2 \geq \theta$ and (3) that

$$h(t, w) \leq h(t_0, \lambda_2) \leq A\lambda_2 \quad \text{on } [0, \sigma(1)] \times [0, \lambda_2].$$

By cases (i) and (ii), if we replace h by f and g , and replace A by A_1 and A_2 , respectively. Then, the hypothesis (1) of Theorem 2.1 is satisfied.

By Remark 2.2, we have the following three corollaries.

Corollary 2.3. *Let*

$$A_i := \left(\int_0^{\sigma(1)} G_i(\sigma(s), s) \Delta s \right)^{-1} \quad \text{and} \quad B_i := \left(\int_{\xi}^{\omega} G_i(\theta, s) \Delta s \right)^{-1}, \quad (i = 1, 2).$$

Then, (1) has at least one positive solution if one of the following conditions holds:

- (H₁) $\max f_0 = D_1 \in [0, A_1)$, $\max g_0 = E_1 \in [0, A_2)$, $\min f_{\infty} = D_2 \in (\frac{B_1}{M_1}, \infty]$ and $\min g_{\infty} = E_2 \in (\frac{B_2}{M_2}, \infty]$;
- (H₂) $\min f_0 = D_3 \in (\frac{B_1}{M_1}, \infty]$, $\min g_0 = E_3 \in (\frac{B_2}{M_2}, \infty]$, $\max f_{\infty} = D_4 \in [0, A_1)$ and $\max g_{\infty} = E_4 \in [0, A_2)$;
- (H₃) $\max f_0 = D_1 \in [0, A_1)$, $\min g_0 = E_3 \in (\frac{B_2}{M_2}, \infty]$, $\min f_{\infty} = D_2 \in (\frac{B_1}{M_1}, \infty]$, and $\max g_{\infty} = E_4 \in [0, A_2)$;
- (H₄) $\min f_0 = D_3 \in (\frac{B_1}{M_1}, \infty]$, $\max g_0 = E_1 \in [0, A_2)$, $\max f_{\infty} = D_4 \in [0, A_1)$ and $\min g_{\infty} = E_2 \in (\frac{B_2}{M_2}, \infty]$.

PROOF. It follows from Remark 2.2 and Theorem 2.1 that the desired result holds, immediately. $\square$

Corollary 2.4. *Let A_i and B_i be defined as in Corollary 2.3. Then, (1) has at least two positive solutions $\mathcal{U}_1$ and $\mathcal{U}_2$ such that*

$$0 < \|\mathcal{U}_1\| < \lambda^* < \|\mathcal{U}_2\|,$$

if the following hypotheses hold:

(H₅) $\min f_\infty = C_1$, $\min f_0 = C_2 \in (\frac{B_1}{M_1}, \infty]$ and $\min g_\infty = D_1$, $\min g_0 = D_2 \in (\frac{B_2}{M_2}, \infty]$;

(H₆) there exists a real number $\lambda^* = \max\{\lambda_1^*, \lambda_2^*\} > 0$ such that

$$\begin{cases} f(t, v) \leq A_1 \lambda_1^* & \text{on } [0, \lambda_1^*], \\ g(t, u) \leq A_2 \lambda_2^* & \text{on } [0, \lambda_2^*]. \end{cases}$$

PROOF. It follows from Remark 2.2 that there exist four real numbers $\eta_{1,1}$, $\eta_{1,2}$, $\eta_{2,1}$ and $\eta_{2,2}$ satisfying

$$0 < \eta_{1,1} < \lambda^* < \eta_{1,2}, \quad 0 < \eta_{2,1} < \lambda^* < \eta_{2,2},$$

$$\begin{cases} f(t, v) \geq B_1 \eta_{1,1} & \text{on } [\xi, \sigma(\omega)] \times [M_1 \eta_{1,1}, \eta_{1,1}], \\ g(t, u) \geq B_2 \eta_{1,2} & \text{on } [\xi, \sigma(\omega)] \times [M_2 \eta_{1,2}, \eta_{1,2}] \end{cases}$$

and

$$\begin{cases} f(t, v) \geq B_1 \eta_{2,1} & \text{on } [\xi, \sigma(\omega)] \times [M_1 \eta_{2,1}, \eta_{2,1}], \\ g(t, u) \geq B_2 \eta_{2,2} & \text{on } [\xi, \sigma(\omega)] \times [M_2 \eta_{2,2}, \eta_{2,2}]. \end{cases}$$

Hence, by Theorem 2.1, we see that (1) has two positive solutions $\mathcal{U}_1$ and $\mathcal{U}_2$ such that

$$\eta_2 < \|\mathcal{U}_1\| < \lambda^* < \|\mathcal{U}_2\| < \eta_1;$$

where $\eta_1 := \min\{\eta_{1,2}, \eta_{2,2}\}$ and $\eta_2 := \min\{\eta_{1,1}, \eta_{2,1}\}$. Thus, we complete the proof. $\square$

Corollary 2.5. *Let A_i and B_i be defined as in Corollary 2.3. Then, (1) has at least two positive solutions $\mathcal{U}_1$ and $\mathcal{U}_2$ such that*

$$0 < \|\mathcal{U}_1\| < \eta^* < \|\mathcal{U}_2\|,$$

if the following hypotheses hold:

(H₇) $\max f_0 = D_1$, $\max f_\infty = D_4 \in [0, A_1)$ and $\max g_0 = E_1$, $\min g_\infty = E_4 \in [0, A_2)$;

(H₈) there exists a real number $\eta^* := \min\{\eta_1^*, \eta_2^*\} > 0$ such that

$$\begin{cases} f(t, v) \geq B_1 \eta_1^* & \text{on } [\xi, \sigma(\omega)] \times [M_1 \eta_1^*, \eta_1^*], \\ g(t, u) \leq B_2 \eta_2^* & \text{on } [\xi, \sigma(\omega)] \times [M_2 \eta_2^*, \eta_2^*]. \end{cases}$$

PROOF. It follows from Remark 2.2, there exist four real numbers $\lambda_{1,1}$, $\lambda_{1,2}$, $\lambda_{2,1}$ and $\lambda_{2,2}$ satisfying

$$0 < \lambda_{1,1} < \eta^* < \lambda_{1,2}, \quad 0 < \lambda_{2,1} < \eta^* < \lambda_{2,2}$$

$$\begin{cases} f(t, v) \leq A_1 \lambda_{1,1} & \text{on } [0, \sigma(1)] \times [0, \lambda_{1,1}], \\ g(t, u) \leq A_2 \lambda_{2,1} & \text{on } [0, \sigma(1)] \times [0, \lambda_{2,1}]. \end{cases}$$

and

$$\begin{cases} f(t, v) \leq A_1 \lambda_{1,2} & \text{on } [0, \sigma(1)] \times [0, \lambda_{1,2}], \\ g(t, u) \leq A_2 \lambda_{2,2} & \text{on } [0, \sigma(1)] \times [0, \lambda_{2,2}]. \end{cases}$$

Hence, by Theorem 2.1 that (1) has two positive solutions $\mathcal{U}_1$ and $\mathcal{U}_2$ such that

$$\lambda_1 < \|\mathcal{U}_1\| < \eta^* < \|\mathcal{U}_2\| < \lambda_2,$$

where $\lambda_1 = \max\{\lambda_{1,1}, \lambda_{2,1}\}$ and $\lambda_2 = \max\{\lambda_{1,2}, \lambda_{2,2}\}$. Thus, we complete the proof. $\square$

Remark 2.6. Consider the following fourth order boundary value problem

$$(BVP.1) \quad \begin{cases} (E_3) & u^{\Delta\Delta\Delta\Delta}(t) + g(t, u(\sigma(t))) = 0, \quad 0 < t < 1, \\ (BC_3) & \begin{cases} \alpha_1 u(0) - \beta_1 u^\Delta(0) = 0, \\ \gamma_1 u(\sigma(1)) + \delta_1 u^\Delta(\sigma(1)) = 0, \end{cases} \\ (BC_4) & \begin{cases} -\alpha_2 u^{\Delta\Delta}(0) + \beta_2 v^{\Delta\Delta\Delta}(0) = 0, \\ -\gamma_2 u^{\Delta\Delta}(\sigma(1)) - \delta_2 u^{\Delta\Delta\Delta}(\sigma(1)) = 0, \end{cases} \end{cases}$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i$ are nonnegative real numbers, $r_i := \gamma_i \beta_i + \alpha_i \delta_i + \alpha_i \gamma_i \sigma(1) > 0$, $i = 1, 2$ and $g \in C_{rd}([0, \sigma(1)] \times [0, \infty), [0, \infty))$. If we let

$-u^{\Delta\Delta}(t) = v(t)$, then (2.6) can be transformed into

$$(BVP.2) \quad \left\{ \begin{array}{ll} (E_4) & u^{\Delta\Delta}(t) + v(t) = 0, \quad 0 < t < 1, \\ (E_5) & v^{\Delta\Delta}(t) + g(t, u(\sigma(t))) = 0, \quad 0 < t < 1, \\ (BC_1) & \begin{cases} \alpha_1 u(0) - \beta_1 u^{\Delta}(0) = 0, \\ \gamma_1 u(\sigma(1)) + \delta_1 u^{\Delta}(\sigma(1)) = 0, \end{cases} \\ (BC_2) & \begin{cases} \alpha_2 v(0) - \beta_2 v^{\Delta}(0) = 0, \\ \gamma_2 v(\sigma(1)) + \delta_2 v^{\Delta}(\sigma(1)) = 0. \end{cases} \end{array} \right.$$

Thus, we can apply the above-mentioned results to study the existence of solutions of (2.6).

References

- [1] R. P. AGARWAL and M. BOHNER, Basic calculus on time scales and some of its applications, *Results Math.* **35** (1999), 3–22.
- [2] R. P. AGARWAL, M. BOHNER and P. J. Y. WONG, Positive solutions and eigenvalues of conjugate boundary value problems, *Proc. Edinburgh Math. Soc.* **42** (1999), 349–374.
- [3] M. BOHNER and A. PETERSON, Dynamic Equations On Time Scales, *Birkhauser, Boston*, 2001.
- [4] C. J. CHYAN and J. HENDERSON, Eigenvalue problems for nonlinear differential equations on a measure chain, *J. Math. Anal. Appl.* **245** (2000), 547–559.
- [5] L. H. ERBE and S. HILGER, Sturm theory on measure chains, *Differential Equations and Dynamical Systems* **1** (1993), 223–246.
- [6] L. ERBE and A. PETERSON, Positive solutions for a nonlinear differential equation on a measure chain, *Math. Comput. Modelling* **32**(5–6) (2000), 571–585.
- [7] L. ERBE and A. PETERSON, Green's functions and comparison theorems for differential equations on measure chains, *Dynam. Contin. Discrete Impuls. Systems* **6** (1999), 121–137.
- [8] S. HILGER, Analysis on measure chains – A unified approach to continuous and discrete calculus, *Results Math.* **18** (1990), 18–56.
- [9] C. H. HONG and C. C. YEH, Positive solutions for eigenvalue problems on a measure chain (*to appear in* *Nonlinear Anal. T. M. & A.*).
- [10] M. A. KRASNOSELSKII, Positive solutions of operator equations, *Noordhoff, Groningen*, 1964.

- [11] V. LAKSHMIKANTHAM, B. KAYMAKALAN and S. SIVASUNDARAM, Dynamic Systems on Measure Chains, *Kluwer Academic Publishers, Boston*, 1996.
- [12] W. C. LIAN, C. C. CHOU, C. T. LIU and F. H. WONG, Existence of solutions of nonlinear BVPs of second order differential equations on measure chains, *Math. & Comp. Mod.* **34** (2001), 821–837.

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(Received July 15, 2002; revised March 20, 2003)