

On the geometry of generalized metric spaces III. Spaces with special forms of curvature tensors

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Dedicated to Professor Lajos Tamásy on his 70th birthday

§0. Introduction

Let M be an n -dimensional differentiable manifold and $T(M)$ its tangent bundle. We consider the bundle M_T which does not contain zero vectors of $T(M)$, that is, $M_T := T(M) - \{0\}$. A generalized metric space M_n is a pair $(M_T, g_{ij}(x, y))$, where $g_{ij}(x, y)$ is a metric tensor satisfying the following assumptions:

- (a) $g_{ij}(x, y)$ is positively homogeneous of degree 0 in y^i ,
- (b) g_{ij} is positive definite,
- (c) $g^*_{ij} := \dot{\partial}_i \dot{\partial}_j F^2 / 2$ is regular, where $\dot{\partial}_i := \partial / \partial y^i$ and $F^2 := g_{ij} y^i y^j$.

In the previous papers ([1],[2]), we introduced three types of connections in M_n : (a) metrical connection $C\Gamma(N) = (F_j^i{}_k, C_j^i{}_k)$, (b) h -metrical connection $R\Gamma(N) = (F_j^i{}_k, 0)$ and (c) non-metrical connection $B\Gamma(G) = (G_j^i{}_k, 0)$ (see [1]).

Two metric tensors g_{ij} , g^*_{ij} are related as

$$g^*_{ij} = g_{ij} + C_{ij}, \quad C_{ij} := y^h \dot{\partial}_j g_{ih} \quad ([1], (2.8)),$$

which satisfies $C_{ij} = C_i^0{}_j = C_{ji}$ and $C_{i0} = 0$, where the index 0 denotes transvection by y .

From the assumption that geodesics introduced from the Finsler metric g^*_{ij} are coincident with those from the generalized metric g_{ij} , that is,

$2G^i = N_0^i$, we have

$$\begin{aligned} N_k^i &= G_k^i - P^i{}_k, \quad G_k^i := \dot{\partial}_k G^i, \\ 2G^i &:= g^{*ih}(y^j \dot{\partial}_h \partial_j F^2 - \partial_h F^2)/2, \end{aligned}$$

where $\partial_j := \partial/\partial x^j$ and the tensor $P^i{}_j$ is arbitrary but $P^i{}_0 = 0$.

In [2], we defined the curvature tensors R , K , H and the generalized metric spaces of R -, K - and H -isotropic (sectional) curvature and obtained the following results.

Theorem A. ([2]) *A generalized metric space of R -isotropic curvature is characterized by*

$$\begin{aligned} (0.1) \quad & 6\{R_{hijk} - R(g_{hj}g_{ik} - g_{hk}g_{ij})\} \\ &= \{(C_{ikr} + 2C_{kir})R^r{}_{hj} + (C_{hjr} + 2C_{jhr})R^r{}_{ik} - j|k\} \\ &+ 2(C_{hir} - C_{ihr})R^r{}_{jk} - (C_{jkr} - C_{kjr})R^r{}_{hi}, \end{aligned}$$

where $j|k$ means the interchange of indices j , k in the foregoing terms.

Theorem B. ([2]) *In a generalized metric space of R -isotropic curvature, if the relation*

$$(0.2) \quad R^i{}_{jk} = R(y_j \delta_k^i - y_k \delta_j^i)$$

is satisfied, then the following equation holds:

$$(0.3) \quad R_h{}^i{}_{jk} = R(g_{hj}\delta_k^i - g_{hk}\delta_j^i).$$

Theorem C. ([2]) *A generalized metric space of K -isotropic curvature is a Riemannian space of constant curvature, that is, $C_{ijk} = 0$ and the following equation holds:*

$$(0.4) \quad K_h{}^i{}_{jk} = K(g_{hj}\delta_k^i - g_{hk}\delta_j^i).$$

Theorem D. ([2]) *A generalized metric space of H -isotropic curvature is a Finsler space of constant curvature, that is, $g^*_{ij} = g_{ij}$ ($C_{ij} = 0$) and the following equation holds:*

$$(0.5) \quad H_h{}^i{}_{jk} = H(g_{hj}\delta_k^i - g_{hk}\delta_j^i).$$

The purpose of the present paper is to consider the inverse problems of the above results. That is, when a generalized metric space has the special forms of curvature tensors: (0.3), (0.4) and (0.5), respectively, we investigate the corresponding properties of the space. These are expressed in Theorems 1.2, 2.1 and 2.5.

We raise or lower the indices by means of g_{ij} only.

Notations and terminologies are those of [1] and [2].

§1. Curvature tensor $H_h{}^i{}_{jk}$

First we shall show

Theorem 1.1. *If a generalized metric space M_n ($n > 2$) satisfies the relation*

$$(1.1) \quad H^i{}_{jk} = Hy_j\delta_k^i - j|k,$$

then the scalar H is a constant, and (1.1) is equivalent to

$$(1.2) \quad H_h{}^i{}_{jk} = H(g_{hj} + C_{hj})\delta_k^i - j|k.$$

PROOF. From (1.1), we have

$$(1.3) \quad (a) \quad H^i{}_{jk} = Hy_jh_k^i - j|k, \quad (b) \quad H^i{}_k = F^2Hh_k^i,$$

where $h_k^i := \delta_k^i - l^i l_k$, $l^i := y^i/F$ and $H^i{}_k := H^i{}_{0k}$.

Substitution of (1.3)(b) into the identity

$$(1.4) \quad 3H^i{}_{jk} = H^i{}_{k(j)} - H^i{}_{j(k)} \quad ([1], (3.6)(b))$$

gives

$$(1.5) \quad H^i{}_{jk} = (Hy_j + \frac{1}{3}F^2H_{(j)})h_k^i - j|k.$$

Comparing (1.5) with (1.3)(a), we get $H_{(j)}h_k^i - j|k = 0$, from which we have $(n-2)H_{(j)} = 0$. Hence, H is independent of y^i .

Next, if we apply the Bianchi identity

$$H^i{}_{jk||l} + j|k|l = 0 \quad ([1], (3.10)(b)),$$

we have

$$H_{||l}(y_jh_k^i - j|k) + H_{||j}(y_kh_l^i - k|l) + H_{||k}(y_lh_j^i - j|l) = 0.$$

Contracting i with l , we get

$$y_jH_{||i}h_k^i - y_kH_{||i}h_j^i = 0.$$

Transvection of this equation by y^j yields $H_{||i}h_k^i = 0$. Since H is independent of y^i , the last equation means

$$(1.6) \quad H_{,k} - (H_{,i}l^i)l_k = 0, \quad H_{,k} := \partial_k H.$$

Differentiating (1.6) by y^j and using $F l^i{}_{(j)} = h_j^i$, $F l_{i(j)} = h_{ij} + C_{ij}$, we have

$$(1.7) \quad (H_{,i}l^i)(g^*{}_{jk} - l_j l_k) = 0.$$

Transvecting (1.7) by g^{*jk} and noting $g^{*jk}l_k = l^j$, we obtain $(n-1)(H_{,i}l^i) = 0$. Making use of this result, we see $H_{,k} = 0$ from (1.6). Hence H is a constant. The last assertion of the theorem is easily derived from $y_{j(h)} = g_{hj} + C_{hj}$. Q.E.D.

Theorem 1.2. *If a generalized metric space M_n ($n > 2$) satisfies the relation*

$$(1.8) \quad H_h{}^i{}_{jk} = H(g_{hj}\delta_k^i - g_{hk}\delta_j^i),$$

then the space is a Finsler space of constant curvature H .

PROOF. It is sufficient to prove $C_{jk} = 0$, which means that the space is a Finsler space. Transvecting (1.8) by y^h and using $H_0{}^i{}_{jk} = H^i{}_{jk}$, we get (1.1). Hence, from Theorem 1.1, we have (1.2). Comparing (1.2) with (1.8), we have $C_{hj}\delta_k^i - j|k = 0$, from which $(n-1)C_{jk} = 0$ is derived. Q.E.D.

§2. Curvature tensors $R_h{}^i{}_{jk}$ and $K_h{}^i{}_{jk}$

Theorem 2.1. *In a generalized metric space M_n ($n > 2$), if the relation*

$$(2.1) \quad R_h{}^i{}_{jk} = R(g_{hj}\delta_k^i - g_{hk}\delta_j^i)$$

is satisfied, then the space is one of R -isotropic curvature.

PROOF. (2.1) gives

$$(2.2) \quad R^i{}_{jk} = R(y_j\delta_k^i - y_k\delta_j^i).$$

It is proved that (2.1) means vanishing of the left-hand side of (0.1) in Theorem A and using (2.2), the right-hand side of (0.1) vanishes. Q.E.D.

It is not yet proved that the scalar R of (2.1) is a constant. Now we shall prepare the following

Proposition 2.2. *In a generalized metric space, the following relations are valid:*

$$(2.3) \quad E^i{}_{k(j)} - E^i{}_{j(k)} = 3E^i{}_{jk} - J_j{}^i{}_{,k}, \quad E^i{}_j := E^i{}_{0j};$$

$$(2.4) \quad R^i{}_{k(j)} - R^i{}_{j(k)} = 3R^i{}_{jk} + J_j{}^i{}_{,k}, \quad R^i{}_j := R^i{}_{0j},$$

where

$$(2.5) \quad (a) \quad J_j{}^i{}_{,k} := P^i{}_{jk/0} + 2(P^i{}_{j/k} + P^i{}_{kr}P^r{}_j + P^i{}_{r}P^r{}_{jk}) - j|k.$$

PROOF. After some calculations, (2.3) follows from

$$(2.6) \quad \begin{aligned} E^i_{jk(h)} &= E^i_{hj} - (P^i_{jh/k} + P^r_{jh}P^i_{kr} - j|k) & ([1], (3.9)(c)), \\ E^i_{hj} + E^i_{kh} + E^i_{hj} &= 0 & ([1], (3.10)(a)). \end{aligned}$$

Next, (2.4) follows from (2.3), (1.4) and

$$(2.7) \quad H^i_{jk} = R^i_{jk} + E^i_{jk}, \quad H^i_k = R^i_k + E^i_k \quad ([2], (1.9)(c), (d)).$$

Q.E.D.

Remark. Using the relation $D^i_{jk} = P^i_{jk} + P^i_{j(k)} = D^i_{kj}$ ([1], (3.2)(a)), we can rewrite (2.5) as

$$(2.5) \quad (b) \quad J^i_{jk} = -P^i_{j(k)/0} + 2(P^i_{j/k} + P^i_{kr}P^r_j + P^i_rP^r_{jk}) - j|k.$$

Theorem 2.3. *In a generalized metric space M_n ($n > 2$) with (2.1), if the tensor J^i_{jk} vanishes, then the scalar R is independent of y^i .*

PROOF. Substituting (2.1) and (2.2) into the relation

$$H^i_{hj} = R^i_{hj} - C^i_{hr}R^r_{jk} + E^i_{hj} \quad ([2], (1.7)(a), (1.9)(b)),$$

we have

$$(2.8) \quad H^i_{hj} = R(g_{hj}\delta^i_k - y_jC^i_{hk} - j|k) + E^i_{hj}.$$

Transvection of (2.8) by y^h gives

$$H^i_{jk} = R(y_j\delta^i_k - j|k) + E^i_{jk}.$$

Differentiating this equation by y^h , we have

$$(2.9) \quad H^i_{hj} = \{R_{(h)}y_j\delta^i_k + R(g_{jh} + C_{jh})\delta^i_k - j|k\} + E^i_{jk(h)}.$$

From (2.8), (2.9) and the identities (2.6), we have

$$R(y_jC^i_{hk} + C_{hj}\delta^i_k) + R_{(h)}y_j\delta^i_k - P^i_{jh/k} - P^r_{jh}P^i_{kr} - j|k = 0.$$

Transvection of this equation by y^j yields

$$\begin{aligned} R(F^2C^i_{hk} - C_{hk}y^i) + F^2R_{(h)}h^i_k - 2P^i_{h/k} + \\ P^i_{kh/0} - 2P^r_hP^i_{kr} + 2P^r_{kh}P^i_r = 0. \end{aligned}$$

Making $-h|k$ in the above equation, we obtain

$$F^2R_{(j)}h^i_k - j|k = J^i_{jk}.$$

Therefore, by our assumption, we get $R_{(j)}h^i_k - j|k = 0$. Contracting i and k , we have $(n-2)R_{(j)} = 0$. Hence R is independent of y^i . Q.E.D.

Theorem 2.4. *In a generalized metric space M_n ($n > 2$) with (2.1), if the tensor P^i_k vanishes, then the scalar R is a constant.*

PROOF. From (2.5)(b), we see that if $P^i_k = 0$, then $J_j^i_k = 0$ holds good. On the other hand, by the definition

$$E^i_{jk} = E_0^i_{jk} = P^i_{j/k} + P^r_j D_r^i_k - j|k \quad ([1], (3.9)(a)),$$

we see that if $P^i_k = 0$, then we have $E^i_{jk} = 0$. Hence, from (2.7), we have $H^i_{jk} = R^i_{jk}$, which means $H^i_{jk} = R y_j \delta_k^i - j|k$ from (2.2). Consequently, noting Theorem 1.1, we have that R is a constant. Q.E.D.

Theorem 2.5. *A generalized metric space M_n ($n > 2$) with*

$$(2.10) \quad K_h^i{}_{jk} = K(g_{hj}\delta_k^i - g_{hk}\delta_j^i)$$

is a Riemannian space of constant curvature.

PROOF. (2.10) is equivalent to $K_{hijk} = K(g_{hj}g_{ik} - g_{hk}g_{ij})$. Consequently, making use of the identity

$$K_{hijk} + K_{ihjk} = -g_{hi(r)} R^r_{jk} \quad ([1], (3.14)(b)),$$

we have

$$(2.11) \quad g_{hi(r)} R^r_{jk} = 0.$$

On the other hand, from (2.10), we see

$$K_h^i{}_{jk} y^h = R^i_{jk} = K(y_j \delta_k^i - y_k \delta_j^i).$$

Substituting this equation into (2.11), we have $g_{hi(k)} y_j - j|k = 0$. Hence, transvection of this equation by y^j gives $g_{hi(k)} = 0$, which means that the space is a Riemannian space of constant curvature. Q.E.D.

References

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