

Note on a Jensen type functional equation

By WILHELMINA SMAJDOR (Kraków)

Abstract. We look for solutions $f : M \rightarrow S$ and examine the stability of the functional equation

$$\begin{aligned} & 3f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) \\ &= 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right], \end{aligned}$$

where M is an Abelian semigroup in which the division by 2 and 3 is performable and S is an abstract convex cone. Some applications to a multivalued version of this equation are given.

1. Introduction

Let $(S, +)$ be an Abelian semigroup, written additively. Suppose that S contains the identity element 0 and for each $\lambda \geq 0$ and $s \in S$, an element λs in S is defined, for which the following axioms hold

$$1s = s, \quad \lambda(\mu s) = (\lambda\mu)s, \quad \lambda(s+t) = \lambda s + \lambda t, \quad (\lambda + \mu)s = \lambda s + \mu s,$$

where $s, t \in S$ and $\lambda, \mu \geq 0$. Then S is said to be an *abstract convex cone*.

If $s, t, t' \in S$, $t + s = t' + s$ always implies that $t = t'$, then S is said to satisfy the *cancellation law*.

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Suppose that an invariant with respect to translations and positively homogeneous metric ϱ is given in S , i.e.,

$$\varrho(t+s, t'+s) = \varrho(t, t')$$

and

$$\varrho(\lambda s, \lambda t) = \lambda \varrho(s, t)$$

for $\lambda > 0$ and $s, t, t' \in S$.

It is easy to see that the mappings $[0, \infty) \times S \ni (\lambda, s) \mapsto \lambda s \in S$ and $S \times S \ni (s, t) \mapsto s+t \in S$ are continuous in the metric topology.

Let $(M, +)$ be an Abelian semigroup with the identity element 0 in which the division by 2 and 3 is performable.

We are going to look for all solutions $f : M \rightarrow S$ of the functional equation

$$\begin{aligned} & 3f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) \\ &= 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right]. \end{aligned} \quad (1)$$

The inequality

$$\begin{aligned} & 3f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) \\ &\geq 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right] \end{aligned}$$

appeared in T. POPOVICIU's paper [3] in connection with the following theorem: The real continuous function f defined on an interval I is convex (i.e. the second divided differences of f are non-negative) if and only if the above inequality holds true for every triples x, y, z in I .

TIBERIU TRIF [7] solved equation (1) in the class of functions $f : X \rightarrow Y$, where X, Y are real vector spaces. His considerations cannot be applied in our reality as subtraction in vector spaces was used.

The main objective of this note is to find all solutions $f : M \rightarrow S$ of (1) and to examine its stability. The natural range of equation (1) is a commutative semigroup. If we consider its stability then the semigroup ought to be endowed with a metric. Restrictions of the second part of

the paper (the range of f is an abstract convex cone with the cancellation law and endowed with a complete metric invariant under translations and positive homogeneous) enable us to prove Theorem 1. The family $\text{clb}(X)$ of all non-empty convex closed and bounded subsets of a real Banach space fulfils these conditions and we can apply Theorem 1 to study equation (1) in the multivalued case. The similar results associated with the Jensen and Pexider functional equations were obtained in [5].

2. Solutions and Hyers–Ulam stability of (1)

We shall assume that

- (i) M is a commutative semigroup with zero in which the division by 2 and 3 is performable;
- (ii) S is an abstract cone satisfying the cancellation law;
- (iii) (S, ρ) is a complete metric space and ρ is invariant with respect to translations and positively homogeneous.

Let $a : M \rightarrow S$ be an additive function which means $a(x + y) = a(x) + a(y)$ for all $x, y \in M$. It is easily seen that for every $b \in S$, the function $f(x) = a(x) + b$, $x \in M$, satisfies (1). The converse follows from the following

Theorem 1. *Assume that conditions (i)–(iii) are fulfilled. If $\varepsilon \geq 0$ and if $f : M \rightarrow S$ satisfies*

$$\begin{aligned} & \varrho \left(3f \left(\frac{x+y+z}{3} \right) + f(x) + f(y) + f(z), \right. \\ & \left. 2 \left[f \left(\frac{x+y}{2} \right) + f \left(\frac{y+z}{2} \right) + f \left(\frac{z+x}{2} \right) \right] \right) \leq \varepsilon \end{aligned} \quad (2)$$

for all $x, y, z \in M$, then there exists a unique additive function $a : M \rightarrow S$ such that

$$\varrho(f(x), a(x) + f(0)) \leq \varepsilon \quad (3)$$

for $x \in M$.

PROOF. Setting in (2) $y = x$ and $z = 0$ we obtain

$$\varrho \left(3f \left(\frac{2}{3}x \right) + 2f(x) + f(0), 2f(x) + 4f \left(\frac{1}{2}x \right) \right) \leq \varepsilon.$$

Since the metric ϱ is invariant with respect to translation and positively homogeneous, we have

$$\varrho\left(\frac{3}{4}f\left(\frac{4}{3}x\right) + \frac{1}{4}f(0), f(x)\right) \leq \frac{1}{4}\varepsilon. \quad (4)$$

Replacing x by $\frac{4}{3}x$, multiplying by $\frac{3}{4}$ both the sides of (4) we infer

$$\varrho\left(\left(\frac{3}{4}\right)^2 f\left(\left(\frac{4}{3}\right)^2 x\right) + \frac{3}{4^2}f(0), \frac{3}{4}f\left(\frac{4}{3}x\right)\right) \leq \frac{3}{4^2}\varepsilon,$$

whence

$$\varrho\left(\left(\frac{3}{4}\right)^2 f\left(\left(\frac{4}{3}\right)^2 x\right) + \frac{1}{4}\left(1 + \frac{3}{4}\right)f(0), \frac{3}{4}f\left(\frac{4}{3}x\right) + \frac{1}{4}f(0)\right) \leq \frac{3}{4^2}\varepsilon.$$

Hence in virtue of (4) follows

$$\varrho\left(\left(\frac{3}{4}\right)^2 f\left(\left(\frac{4}{3}\right)^2 x\right) + \frac{1}{4}\left(1 + \frac{3}{4}\right)f(0), f(x)\right) \leq \frac{1}{4}\left(1 + \frac{3}{4}\right)\varepsilon.$$

By induction we can show that for every positive integer n the inequality

$$\begin{aligned} \varrho\left(\left(\frac{3}{4}\right)^n f\left(\left(\frac{4}{3}\right)^n x\right) + \frac{1}{4}\left[1 + \frac{3}{4} + \cdots + \left(\frac{3}{4}\right)^{n-1}\right]f(0), f(x)\right) \\ \leq \frac{1}{4}\left(1 + \frac{3}{4} + \cdots + \left(\frac{3}{4}\right)^{n-1}\right)\varepsilon \end{aligned} \quad (5)$$

holds. Write

$$f_n(x) := \left(\frac{3}{4}\right)^n f\left(\left(\frac{4}{3}\right)^n x\right), \quad x \in M, \quad n \in \mathbb{N}. \quad (6)$$

It follows by (5) that for arbitrary $n, m \in \mathbb{N}$, and $x \in M$ we have

$$\begin{aligned} \varrho(f_{n+m}(x), f_n(x)) &= \varrho\left(\left(\frac{3}{4}\right)^{n+m} f\left(\left(\frac{4}{3}\right)^{n+m} x\right), \left(\frac{3}{4}\right)^n f\left(\left(\frac{4}{3}\right)^n x\right)\right) \\ &= \left(\frac{3}{4}\right)^n \varrho\left(\left(\frac{3}{4}\right)^m f\left(\left(\frac{4}{3}\right)^{n+m} x\right) + \frac{1}{4}\left[1 + \frac{3}{4} + \cdots + \left(\frac{3}{4}\right)^{m-1}\right]f(0), \right. \end{aligned}$$

$$\begin{aligned}
& f\left(\left(\frac{4}{3}\right)^n x\right) + \frac{1}{4} \left[1 + \frac{3}{4} + \cdots + \left(\frac{3}{4}\right)^{m-1}\right] f(0) \\
& \leq \left(\frac{3}{4}\right)^n \varrho\left(\left(\frac{3}{4}\right)^m f\left(\left(\frac{4}{3}\right)^m \left(\frac{4}{3}\right)^n x\right)\right. \\
& \quad \left. + \frac{1}{4} \left[1 + \frac{3}{4} + \cdots + \left(\frac{3}{4}\right)^{m-1}\right] f(0), f\left(\left(\frac{4}{3}\right)^n x\right)\right) \\
& \quad + \left(\frac{3}{4}\right)^n \varrho\left(f\left(\left(\frac{4}{3}\right)^n x\right), f\left(\left(\frac{4}{3}\right)^n x\right)\right) \\
& \quad + \frac{1}{4} \left[1 + \frac{3}{4} + \cdots + \left(\frac{3}{4}\right)^{m-1}\right] f(0) \\
& \leq \left(\frac{3}{4}\right)^n \frac{1}{4} \left[1 + \frac{3}{4} + \cdots + \left(\frac{3}{4}\right)^{m-1}\right] \varepsilon \\
& \quad + \left(\frac{3}{4}\right)^n \frac{1}{4} \left[1 + \frac{3}{4} + \cdots + \left(\frac{3}{4}\right)^{m-1}\right] \varrho(0, f(0)) \\
& < \left(\frac{3}{4}\right)^n [\varepsilon + \varrho(0, f(0))].
\end{aligned}$$

Thus for every $x \in M$, $(f_n(x))_{n \in \mathbb{N}}$ is a Cauchy sequence. Let

$$a(x) = \lim_{n \rightarrow \infty} f_n(x), \quad x \in M. \quad (7)$$

Letting $n \rightarrow \infty$, we obtain by (5)

$$\varrho(a(x) + f(0), f(x)) \leq \varepsilon.$$

We have by (2)

$$\begin{aligned}
& \varrho\left(3\left(\frac{3}{4}\right)^n f\left(\frac{1}{3}\left[\left(\frac{4}{3}\right)^n x + \left(\frac{4}{3}\right)^n y + \left(\frac{4}{3}\right)^n z\right]\right)\right. \\
& \quad \left. + \left(\frac{3}{4}\right)^n f\left(\left(\frac{4}{3}\right)^n x\right) + \left(\frac{3}{4}\right)^n f\left(\left(\frac{4}{3}\right)^n y\right) + \left(\frac{3}{4}\right)^n f\left(\left(\frac{4}{3}\right)^n z\right),\right. \\
& \quad 2\left[\left(\frac{3}{4}\right)^n f\left(\left(\frac{4}{3}\right)^n \frac{x+y}{2}\right) + \left(\frac{3}{4}\right)^n f\left(\left(\frac{4}{3}\right)^n \frac{y+z}{2}\right)\right. \\
& \quad \left. \left. + \left(\frac{3}{4}\right)^n f\left(\left(\frac{4}{3}\right)^n \frac{z+x}{2}\right)\right]\right) \leq \left(\frac{3}{4}\right)^n \varepsilon,
\end{aligned}$$

i.e.,

$$\varrho \left(3f_n \left(\frac{x+y+z}{3} \right) + f_n(x) + f_n(y) + f_n(z), \right. \\ \left. 2 \left[f_n \left(\frac{x+y}{2} \right) + f_n \left(\frac{y+z}{2} \right) + f_n \left(\frac{z+x}{2} \right) \right] \right) \leq \left(\frac{3}{4} \right)^n \varepsilon.$$

Passing to the limit as $n \rightarrow \infty$ we get

$$3a \left(\frac{x+y+z}{3} \right) + a(x) + a(y) + a(z) \\ = 2 \left[a \left(\frac{x+y}{2} \right) + a \left(\frac{y+z}{2} \right) + a \left(\frac{z+x}{2} \right) \right] \quad (8)$$

for $x, y, z \in M$, which means that a satisfies equation (1). Since $a(0) = \lim_{n \rightarrow \infty} (3/4)^n f(0)$ by (7) and (6), we have $a(0) = 0$. Now we shall prove that a is an additive function. Putting in (8) $y = x$ we get

$$3a \left(\frac{2x+z}{3} \right) + a(z) = 4a \left(\frac{x+z}{2} \right). \quad (9)$$

If we put $u = \frac{2x+z}{3}$, (9) turns into

$$3a(u) + a(z) = 4a \left(\frac{3u+z}{4} \right), \quad u, z \in M. \quad (10)$$

Substitute $z = 0$ in (10). Then $a \left(\frac{3}{4}u \right) = \frac{3}{4}a(u)$ or

$$a(3u) = \frac{3}{4}a(4u). \quad (11)$$

Setting $u = 0$ in (10) we obtain $a(z) = 4a \left(\frac{1}{4}z \right)$ or $a(4z) = 4a(z)$. Hence and by (11), $a(3u) = 3a(u)$. Now formula (10) may be rewritten in the form $a(3u) + a(z) = a(3u + z)$, whence the additivity of a follows.

To end the proof we have to show the uniqueness of a .

Suppose that (3) holds with an additive function $\tilde{a} : M \rightarrow S$. We have for arbitrary $n \in \mathbb{N}$

$$\varrho(a(x), \tilde{a}(x)) = \left(\frac{3}{4} \right)^n \varrho \left(\left(\frac{4}{3} \right)^n a(x), \left(\frac{4}{3} \right)^n \tilde{a}(x) \right)$$

$$\begin{aligned}
 &= \left(\frac{3}{4}\right)^n \varrho \left(a \left(\left(\frac{4}{3}\right)^n x \right), \tilde{a} \left(\left(\frac{4}{3}\right)^n x \right) \right) \\
 &= \left(\frac{3}{4}\right)^n \varrho \left(a \left(\left(\frac{4}{3}\right)^n x \right) + f(0), \tilde{a} \left(\left(\frac{4}{3}\right)^n x \right) + f(0) \right) \\
 &\leq \left(\frac{3}{4}\right)^n \varrho \left(a \left(\left(\frac{4}{3}\right)^n x \right) + f(0), f \left(\left(\frac{4}{3}\right)^n x \right) \right) \\
 &\quad + \left(\frac{3}{4}\right)^n \varrho \left(f \left(\left(\frac{4}{3}\right)^n x \right), \tilde{a} \left(\left(\frac{4}{3}\right)^n x \right) + f(0) \right) \leq 2 \left(\frac{3}{4}\right)^n \varepsilon,
 \end{aligned}$$

whence $a(x) = \tilde{a}(x)$, $x \in M$. $\square$

Taking $\varepsilon = 0$ in Theorem 1 we obtain the following

Theorem 2. *Assume that conditions (i)–(iii) are fulfilled. If $f : M \rightarrow S$ satisfies (1), then there exists an additive function $a : M \rightarrow S$ and $b \in S$ such that $f(x) = a(x) + b$, $x \in M$.*

3. Multivalued solutions of (1)

Let X be a real Banach space and let $\text{clb}(X)$ denote the set of all non-empty convex closed and bounded subsets of X . Introduce a binary operation $+$ in $\text{clb}(X)$ by the formula

$$A +^* B = \text{cl}(A + B) = \text{cl}(\text{cl } A + \text{cl } B),$$

where $A + B$ denotes the usual Minkowski sum of A and B while $\text{cl } A$ denotes the closedness of the set A . The second operation in $\text{clb}(X)$ is given by

$$\lambda A = \{\lambda a : a \in A\}$$

for all $\lambda \geq 0$ and $A \in \text{clb}(X)$. It is easily seen that $\text{clb}(X)$ is an abstract convex cone with the identity element $0 := \{0\}$.

The proof of the following generalization of the RÅDSTRÖM lemma (cf. [4]) can be found in [6].

Lemma 1. *If a set $B \subset X$ is a non-empty and bounded and $C \subset X$ is convex and closed, then for every $A \subset X$,*

$$A + B \subset C +^* B \implies A \subset C.$$

From Lemma 1, we derive that the cancellation law holds in the abstract convex cone $\text{clb}(X)$.

The set $\text{clb}(X)$ is a metric space with the Hausdorff distance h defined as follows

$$h(A, B) = \max \left\{ \sup \{d(a, B) : a \in A\}, \sup \{d(b, A) : b \in B\} \right\},$$

where $d(a, B) = \inf \{\|a - b\| : b \in B\}$. The metric space $(\text{clb}(X), h)$ is complete (cf. e.g. [1]).

Lemma 2. *If $A, B, C \in \text{clb}(X)$ and $\lambda \geq 0$, then*

$$h(A \overset{*}{+} B, C \overset{*}{+} B) = h(A + B, C + B) = h(A, C), \quad (12)$$

and

$$h(\lambda A, \lambda B) = \lambda h(A, B). \quad (13)$$

The first equality in (12) is easy to verify, the proof of the second one can be found in [2]. Formula (13) is well known. Thus the abstract cone $\text{clb}(X)$ satisfy assumptions (ii) and (iii).

A multifunction $F_0 : M \rightarrow \text{clb}(X)$ is said to be $(*)$ -additive if

$$F_0(x + y) = F_0(x) \overset{*}{+} F_0(y)$$

for all $x, y \in M$.

From Theorem 1 we derive the following result.

Theorem 3. *Let $(M, +)$ satisfy condition (i) and let X be a real Banach space. We assume that $\varepsilon \geq 0$ and that $F : M \rightarrow \text{clb}(X)$ satisfies the inequality*

$$h \left(3F \left(\frac{x + y + z}{3} \right) \overset{*}{+} F(x) \overset{*}{+} F(y) \overset{*}{+} F(z), \right. \\ \left. 2 \left[F \left(\frac{x + y}{2} \right) \overset{*}{+} F \left(\frac{y + z}{2} \right) \overset{*}{+} F \left(\frac{z + x}{2} \right) \right] \right) \leq \varepsilon,$$

then there exists a unique $(*)$ -additive multifunction $F_0 : M \rightarrow \text{clb}(X)$ such that

$$h \left(F(x), F_0(x) \overset{*}{+} F(0) \right) \leq \varepsilon$$

for all $x, y \in M$.

In particular, putting $\varepsilon = 0$, we have the following result.

Theorem 4. *Let $(M, +)$ satisfy assumption (i) and let X be a real Banach space. If $F : M \rightarrow \text{clb}(X)$ satisfies the functional equation*

$$\begin{aligned} & 3F\left(\frac{x+y+z}{3}\right) \overset{*}{+} F(x) \overset{*}{+} F(y) \overset{*}{+} F(z) \\ &= 2 \left[F\left(\frac{x+y}{2}\right) \overset{*}{+} F\left(\frac{y+z}{2}\right) \overset{*}{+} F\left(\frac{z+x}{2}\right) \right], \end{aligned} \quad (14)$$

then there exists a $(*)$ -additive multifunction $F_0 : M \rightarrow \text{clb}(X)$ such that

$$F(x) = F_0(x) \overset{*}{+} F(0)$$

for all $x \in M$. Conversely, every multifunction $F(x) = F_0(x) \overset{*}{+} B$, where $F_0 : M \rightarrow \text{clb}(X)$ is a $(*)$ -additive multifunction and $B \in \text{clb}(X)$ is an arbitrary set, actually satisfies (14).

Remark 1. Every additive multifunction $F_0 : M \rightarrow \text{clb}(X)$ which means

$$F_0(x+y) = F_0(x) + F_0(y) \quad \text{for all } x, y \in M \quad (15)$$

is $(*)$ -additive. In fact by (15), $F_0(x+y) = \text{cl}(F_0(x+y)) = \text{cl}(F_0(x) + F_0(y)) = F_0(x) \overset{*}{+} F_0(y)$.

Remark 2. A $(*)$ -additive multifunction $F_0 : M \rightarrow \text{clb}(X)$ does not have to be additive. To see that take $A, B \in \text{clb}(X)$ such that $\text{cl}(A+B) \neq A+B$. The authoress believes that an example such sets A, B is known but we will construct one below for convenience of a reader. The multifunction $F : [0, \infty)^2 \rightarrow \text{clb}(X)$ given by the formula

$$F(t_1, t_2) = \text{cl}(t_1A + t_2B)$$

is $(*)$ -additive. Indeed,

$$\begin{aligned} F((t_1, t_2) + (s_1, s_2)) &= F(t_1 + s_1, t_2 + s_2) = \text{cl}[(t_1 + s_1)A + (t_2 + s_2)B] \\ &= \text{cl}[t_1A + t_2B + s_1A + s_2B] = \text{cl}[\text{cl}(t_1A + t_2B) + \text{cl}(s_1A + s_2B)] \\ &= F(t_1, t_2) \overset{*}{+} F(s_1, s_2) \end{aligned}$$

for all $t_1, t_2, s_1, s_2 \in [0, \infty)$. However, F is not additive, as

$$F(1, 0) + F(0, 1) = \text{cl } A + \text{cl } B = A + B$$

and

$$F((1, 0) + (0, 1)) = F(1, 1) = \text{cl}(A + B) \neq A + B = F(1, 0) + F(0, 1).$$

The following example has been suggested by Dr ANNA KUCIA (Katowice), the authoress wish to thank her for that in this place. Let $X = l_1$ denote the space of all summable sequences real numbers. For each $i \in \mathbb{N}$, let e_i be the vector in l_1 with zeros in all its coordinates except the i^{th} coordinate which is equal to one. Define

$$A_1 = \left\{ \left(1 + \frac{1}{i}\right) e_i : i \in \mathbb{N} \right\}, \quad B_1 = \left\{ \left(-1 + \frac{1}{i}\right) e_i : i \in \mathbb{N} \right\},$$

and

$$A = \overline{\text{co}} A_1, \quad B = \overline{\text{co}} B_1,$$

where $\overline{\text{co}} A_1$ denotes the intersection of all convex closed sets containing A_1 . At first we observe that

$$A_0 := \left\{ \left(2p_1, \frac{3}{2}p_2, \frac{4}{3}p_3, \dots, \frac{i+1}{i}p_i, \dots\right) : \sum_{i=1}^{\infty} p_i = 1, p_i \geq 0 \right\}$$

$$\subset A = \text{cl}(\text{co } A_1);$$

for every element of A_0 is a limit of some sequence of points belonging to $\text{co } A_1$.

Next, we shall show that $A \subset A_0$. Take an arbitrary $a = (2p_1, \frac{3}{2}p_2, \frac{4}{3}p_3, \dots, \frac{i+1}{i}p_i, \dots) \in A$. Of course $p_i \geq 0$, $i \in \mathbb{N}$. It is enough to prove that $\sum_{i=1}^{\infty} p_i = 1$. We can find $a^n = (2p_1^n, \frac{3}{2}p_2^n, \frac{4}{3}p_3^n, \dots, \frac{i+1}{i}p_i^n, \dots) \in \text{co } A_1$, $n \in \mathbb{N}$ such that $\lim_{n \rightarrow \infty} a^n = a$ and $r_m^n := 1 - \sum_{i=1}^m p_i^n \rightarrow 0$ as $m \rightarrow \infty$ for each $n \in \mathbb{N}$. Since

$$\frac{i+1}{i} |p_i^n - p_i| \leq \sum_{i=1}^{\infty} \frac{i+1}{i} |p_i^n - p_i| = \|a^n - a\|$$

and $\lim_{n \rightarrow \infty} \|a^n - a\| = 0$,

$$\sum_{i=1}^m p_i^n \rightarrow \sum_{i=1}^m p_i \quad \text{as } n \rightarrow \infty$$

for every $m \in \mathbb{N}$. Thus

$$\lim_{n \rightarrow \infty} r_m^n = 1 - \sum_{i=1}^m p_i =: r_m \quad (16)$$

exists for each $m \in \mathbb{N}$. Now we shall show that the sequence $(r_m^n)_{n \in \mathbb{N}}$ satisfies the Cauchy condition uniformly with respect to m . We have for all $m, k, n \in \mathbb{N}$,

$$|r_m^n - r_m^k| = \left| \sum_{i=1}^m p_i^k - \sum_{i=1}^m p_i^n \right| \leq \sum_{i=1}^m |p_i^k - p_i^n| \leq \sum_{i=1}^{\infty} |p_i^k - p_i^n|.$$

Let us fix $\varepsilon > 0$. Since (a^n) is convergent, there exists a positive number α such that

$$\sum_{i=1}^{\infty} \frac{i+1}{i} |p_i^k - p_i^n| < \varepsilon$$

for all $n, k > \alpha$. Hence

$$|r_m^n - r_m^k| < \varepsilon, \quad n, k > \alpha \quad \text{and} \quad m \in \mathbb{N}.$$

Now, when $k \rightarrow \infty$, the sequence $(r_m^k)_{k \in \mathbb{N}}$ tends to r_m and

$$|r_m^n - r_m| \leq \varepsilon, \quad \text{for } n > \alpha \quad \text{and } m \in \mathbb{N}.$$

Fix arbitrarily $n > \alpha$. Since for every $m \in \mathbb{N}$,

$$0 \leq r_m \leq |r_m - r_m^n| + r_m^n,$$

letting $m \rightarrow \infty$, we obtain $\limsup_{m \rightarrow \infty} r_m \leq \varepsilon$. Consequently, $\lim_{m \rightarrow \infty} r_m = 0$ and by (16), $\sum_{i=1}^{\infty} p_i = 1$. We have proved that $A = A_0$. Similarly we can show that

$$B = \left\{ \left(0, -\frac{1}{2}q_2, -\frac{2}{3}q_3, \dots, -\frac{i-1}{i}q_i, \dots \right) : \sum_{i=2}^{\infty} q_i = 1, q_i \geq 0 \right\}.$$

We observe that $0 \in \text{cl}(A+B)$. Indeed the set $A+B$ contains points of the form $2 \left(p_1, \frac{1}{2}p_2, \dots, \frac{1}{i}p_i, \dots \right)$, where $p_i \geq 0$ and $\sum_{i=1}^{\infty} p_i = 1$. Thus this set has elements of arbitrarily small norms. To show that $0 \notin A+B$ we argue by contradiction: if for each $i \in \mathbb{N}$, $\frac{i+1}{i}p_i - \frac{i-1}{i}q_i = 0$ and $\sum_{i=1}^{\infty} p_i = 1$, $\sum_{i=1}^{\infty} q_i = 1$, we would have then

$$1 = \sum_{i=1}^{\infty} p_i = \sum_{i=1}^{\infty} \frac{i-1}{i+1} q_i = 1 - 2 \sum_{i=1}^{\infty} \frac{1}{i+1} q_i < 1.$$

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WILHELMINA SMAJDOR
INSTITUTE OF MATHEMATICS
PEDAGOGICAL UNIVERSITY
PODCHORĄŻYCH 2, PL-30-084 KRAKÓW
POLAND

E-mail: wsmajdor@wsp.krakow.pl

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