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# On varieties defined by pseudocomplemented nondistributive lattices

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**Abstract.** Lattices with 1, where for each element $a \in L$ the interval $[a, 1]$ is pseudocomplemented, can be equipped with a binary operation "$\circ$" similar to the operation of relative pseudocomplementation. These algebras $(L, \wedge, \vee, \circ, 1)$ form an arithmetical and 1-regular variety. We investigate the subvarieties and the congruence kernels in this variety. It is shown that all algebras $(L, \wedge, \vee, \circ, 1)$ where $L$ is a finite sublattice of a free lattice can be characterized by a particular identity.

## 1. Introduction

A bounded lattice $L$ is called *pseudocomplemented* if for any $x \in L$ there exists an element $x^* \in L$ with the property that

$$y \wedge x = 0 \quad \text{if and only if } y \leqq x^*.$$

In [4] were characterized lattices with greatest element 1 where for each element $a \in L$ the interval $[a, 1]$ is pseudocomplemented. It was shown that they can be equipped with a binary operation "$\circ$"having similar properties as the operation of relative pseudocomplementation and that the class $\mathcal{P}$ of all these algebras $(L, \wedge, \vee, \circ, 1)$ is equational. Although lattices with relative pseudocomplementation are always distributive (see e.g. [2]), the

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above mentioned operation $\circ$ can be defined for nondistributive lattices, too. These preliminary results are discussed in Section 2.

An important subclass of the class of relatively pseudocomplemented lattices are *relative*$(L_n)$*-lattices* introduced in [11]. In Section 3, by the mean of the operation $\circ$, this notion is successfully extended to the nondistributive case. As a result we obtain new subvarieties of the variety $\mathcal{P}$. In Section 4 we apply our results to finite sublattices of free lattices. In Section 5 the congruence properties of $\mathcal{P}$ are investigated and we characterize the congruence kernels in the algebras of $\mathcal{P}$.

## 2. Preliminaries

Let $L$ be a lattice with 1 and $x, y \in L$. The pseudocomplement of $x \vee y$ in the interval $[y, 1]$ (if it exists) is denoted by $x \circ y$ (see [4]), and it is called the *section pseudocomplement of $x$ with respect to $y$*.

**Lemma 2.1.** *The following conditions are equivalent for a lattice $L$ with 1.*

(a) *For any $x, y \in L$ the section pseudocomplement $x \circ y$ exists in $L$.*

(b) *Any principal filter $[a, 1]$ of $L$ is a pseudocomplemented lattice.*

(c) *For any $a \leqq b$ the interval $[a, b]$ is a pseudocomplemented lattice.*

Proof. The equivalence of (a) and (b) was proved in [4] and (c) $\implies$ (b) is clear. As any principal ideal of a pseudocomplemented lattice is also a pseudocomplemented lattice, (b) $\implies$ (c) is obvious. $\square$

A lattice $L$ with 1 is called *sectionally pseudocomplemented*, if the section pseudocomplement $x \circ y$ exists for each $x, y \in L$.

*Remark 2.2.* We recall that $(L, \wedge, \vee, *, 1)$ is a *Brouwerian algebra (or a relatively pseudocomplemented lattice)* if $(L, \wedge, \vee, 1)$ is a lattice with 1 and having the property that for any $a, b, x \in L$,

$$a \wedge x \leqq b \iff x \leqq a * b.$$

The operation $\circ$ can be considered as an extension of $*$, since for $x \in [y, 1]$ we have $x * y = x \circ y$, whenever $x * y$ exists (see [4]). If $L$ is a distributive lattice, then the operations $\circ$ and $*$ coincide (see [1]).

*Example 2.3.* The lattice $N_5$ (see Figure 1) is sectionally pseudocomplemented but not relatively pseudocomplemented (see [4]).

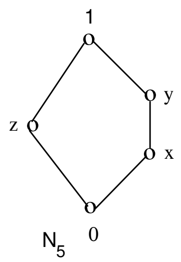


*Figure 1*

E.g. the relative pseudocomplement $y * x$ does not exist, however the section pseudocomplement $y \circ x$ exists and equals to $x$.

A lattice $L$ is $\wedge$-*semidistributive* if $a \wedge b_1 = a \wedge b_2$ implies $a \wedge b_1 = a \wedge (b_1 \vee b_2)$ for any $a, b_1, b_2 \in L$. A complete lattice $L$ is called *completely* $\wedge$-*semidistributive* if for any $b_i \in L$, $i \in I$ and $a \in L$ the relations $a \wedge b_i = y$, $i \in I$ imply $a \wedge (\bigvee\{b_i \mid i \in I\}) = y$. In view of [4], for any complete lattice $L$ the conditions of the Lemma 2.1 are equivalent to the condition

(d) $L$ is completely $\wedge$-semidistributive.

A lattice $L$ with 0 is called 0-*distributive* if for any elements $a, b, c \in L$ $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$. Clearly, any principal filter of a $\wedge$-semidistributive lattice is 0-distributive. According to [14; Theorem 1], an algebraic lattice is pseudocomplemented if and only if it is 0-distributive. These results leads us to the following

**Proposition 2.4.** *If $L$ is an algebraic lattice, then the following conditions are equivalent:*

(i) *$L$ is sectionally pseudocomplemented.*

(ii) *$L$ is completely $\wedge$-semidistributive.*

(iii) *$L$ is $\wedge$-semidistributive.*

(iv) *Any principal filter $[a, 1]$ of $L$ is a 0-distributive lattice.*

Proof. The implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are obvious and the equivalence (i) $\Longleftrightarrow$ (ii) was established in [4]. As any principal filter

$[a, 1]$ of an algebraic lattice is also an algebraic lattice, the above mentioned result of [14] gives (iv) $\Longrightarrow$ (b), whence using Lemma 2.1 we get (iv) $\Longrightarrow$ (i). $\square$

We note that in the rest of the paper we deal with arbitrary sectionally pseudocomplemented lattices and in general we do not assume that they are algebraic or complete.

Let $\mathcal{P}$ denote the class of all algebras $(L, \wedge, \vee, \circ, 1)$, defined on sectionally pseudocomplemented lattices $(L, \wedge, \vee, 1)$. In [4] it was shown that the class $\mathcal{P}$ is determined by identities in signature $\{\wedge, \vee, \circ, 1\}$, namely by the lattice axioms and by the identities

(1) $x \circ x = 1,\ 1 \circ x = x$

(2) $((x \circ y) \circ y) \wedge (x \vee y) = x \vee y$

(3) $(x \vee y) \circ y = x \circ y,\ \ y \vee (x \circ y) = x \circ y$

(4) $([(x \vee z) \wedge (y \vee z)] \circ z) \wedge ([(x \vee z) \wedge (y \circ z)] \circ z) = x \circ z$

Thus $\mathcal{P}$ is a variety and, according to Remark 2.3, $\mathcal{P}$ contains as a subvariety the variety $\mathcal{B}$ of all Brouwerian algebras.

## 3. Hereditary weakly $L_n$-lattices

*Definition 3.1.* (i) Let $L$ be a pseudocomplemented lattice and $n \geqq 1$. We say that $L$ is a *weakly $L_n$-lattice,* if it satisfies the equation:

$$(x_1 \wedge \ldots \wedge x_n)^* \vee (x_1^* \wedge \ldots \wedge x_n)^* \vee \ldots \vee (x_1 \wedge \ldots \wedge x_n^*)^* = 1. \qquad (L_n)$$

If in addition $L$ is distributive, then it is called an *$(L_n)$-lattice* [11].

(ii) $L$ is called a *hereditary weakly $(L_n)$-lattice* if any principal filter $[a, 1]$ of it is a weakly $(L_n)$-lattice.

Notice, that for $n = 1$ Definition 3.1(i) gives $x^* \vee x^{**} = 1$ and we say that the lattice $L$ is *weakly Stonean.* If $L$ is a distributive lattice, then Definition 3.1(ii) implies that any interval $[a, b] \subseteqq L$ is also an $(L_n)$-lattice. Lattices with this property were called in [11] *relative $(L_n)$-lattices.* (Relative $(L_1)$-lattices are known also under the name *relative Stone* lattices, see e.g. [10].)

In [11] M. Haviar and T. Katriňák established that any (distributive) relative $(L_n)$-lattice is characterized by the equation

$$(x_1\wedge\ldots\wedge x_n)*y\vee(x_1*y\wedge\ldots\wedge x_n)*y\vee\ldots\vee(x_1\wedge\ldots\wedge x_n*y)*y=1. \quad (L_n)'$$

We shall deduce a similar equation with $\circ$ for hereditary weakly $(L_n)$-lattices. The lemma below follows directly from the definition of a section pseudocomplemented lattice:

**Lemma 3.2.** *If $L$ is a sectionally pseudocomplemented lattice, then $x \leqq z \implies x\circ y \geqq z\circ y$.*

**Theorem 3.3.** *Let $L$ be a lattice with 1. Then the following assertions are equivalent:*

(i) *$L$ is a hereditary weakly $(L_n)$-lattice.*

(ii) *$x\circ y$ is defined for all $x,y\in L$ and the algebra $(L,\wedge,\vee,\circ,1)$ satisfies the equation:*

$$(x_1\wedge\ldots\wedge x_n)\circ y\vee(x_1\circ y\wedge\ldots\wedge x_n)\circ y\vee\ldots\vee(x_1\wedge\ldots\wedge x_n\circ y)\circ y=1. \quad (P_n)$$

Proof. (i) $\implies$ (ii): As $L$ is a hereditary weakly $(L_n)$-lattice, any principal filter $[y)$ of it is pseudocomplemented, therefore, in virtue of Lemma 2.1, $x\circ y$ is defined for all $x,y\in L$. Since $x_1\wedge\ldots\wedge x_n \leqq (x_1\vee y)\wedge\ldots\wedge(x_n\vee y)$, Lemma 3.2 gives $(x_1\wedge\ldots\wedge x_n)\circ y \geqq [(x_1\vee y)\wedge\ldots\wedge(x_n\vee y)]\circ y$.

On the other hand, using the identity $(x\vee y)\circ y = x\circ y$ we get

$$x_1\circ y\wedge\ldots\wedge x_n \leqq (x_1\vee y)\circ y\wedge\ldots\wedge(x_n\vee y),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_1\wedge\ldots\wedge x_n\circ y \leqq (x_1\vee y)\wedge\ldots\wedge(x_n\vee y)\circ y.$$

By applying Lemma 3.2 again, we deduce

$$\begin{aligned}(x_1\wedge&\ldots\wedge x_n)\circ y\vee(x_1\circ y\wedge\ldots\wedge x_n)\circ y\vee\ldots\vee(x_1\wedge\ldots\wedge x_n\circ y)\circ y\\ &\geqq[(x_1\vee y)\wedge\ldots\wedge(x_n\vee y)]\circ y\vee[(x_1\vee y)\circ y\wedge\ldots\wedge(x_n\vee y)]\circ y\vee\ldots\\ &\vee[(x_1\vee y)\wedge\ldots\wedge(x_n\vee y)\circ y]\circ y.\end{aligned}$$

Since $x_1\vee y\geqq y,\ldots,\ x_n\vee y\geqq y$, and $(x_1\vee y)\circ y\geqq y,\ldots,(x_n\vee y)\circ y\geqq y$, all these elements belong to the pseudocomplemented lattice $[y,1]$. Let $u^y$

denote the pseudocomplement of an element $u \geqq y$ in the lattice $[y, 1]$. Now, in view of definition of the operation $\circ$ we obtain:

$$[(x_1 \vee y) \wedge \ldots \wedge (x_n \vee y)] \circ y = [(x_1 \vee y) \wedge \ldots \wedge (x_n \vee y)]^y,$$

$$[(x_1 \vee y) \circ y \wedge \ldots \wedge (x_n \vee y)] \circ y = [(x_1 \vee y)^y \wedge \ldots \wedge (x_n \vee y)]^y,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$[(x_1 \vee y) \wedge \ldots \wedge (x_n \vee y) \circ y] \circ y = [(x_1 \vee y) \wedge \ldots \wedge (x_n \vee y)^y]^y.$$

Summarizing the above results, and taking in consideration that by assumption the lattice $[y, 1]$ satisfies the identity $(L_n)$ we obtain:

$$\begin{aligned}(x_1 \wedge \ldots \wedge x_n) \circ y \vee (x_1 \circ y \wedge \ldots \wedge x_n) \circ y \vee \ldots \vee (x_1 \wedge \ldots \wedge x_n \circ y) \circ y \\ \geqq [(x_1 \vee y) \wedge \ldots \wedge (x_n \vee y)]^y \vee [(x_1 \vee y)^y \wedge \ldots \wedge (x_n \vee y)]^y \vee \ldots \\ \vee [(x_1 \vee y) \wedge \ldots \wedge (x_n \vee y)^y]^y = 1.\end{aligned}$$

Hence $(P_n)$ holds in $L$ and this proves (ii).

(ii) $\implies$ (i): Assume that $(P_n)$ holds in the algebra $(L, \wedge, \vee, \circ, 1)$ and take an $a \in L$. As $x \circ y$ is defined for all $x, y \in L$, in view of Lemma 2.1 the interval $[a, 1]$ is a pseudocomplemented lattice. Let $x_1, \ldots, x_n \geqq a$. Since for any $x \in [a, 1]$ we have $x \circ a = x^a$, we get

$$\begin{aligned}(x_1 \wedge \ldots \wedge x_n)^a \vee (x_1^a \wedge \ldots \wedge x_n)^a \vee \ldots \vee (x_1 \wedge \ldots \wedge x_n^a)^a \\ = (x_1 \wedge \ldots \wedge x_n) \circ a \vee (x_1 \circ a \wedge \ldots \wedge x_n) \circ a \vee \ldots \\ \vee (x_1 \wedge \ldots \wedge x_n \circ a) \circ a = 1.\end{aligned}$$

This equation shows that for any $a \in L$, the principal filter $[a)$ is an $(L_n)$-lattice. Thus $L$ is a hereditary weakly $(L_n)$-lattice. □

*Example 3.4.* The algebra $(N_5, \wedge, \vee, \circ, 1)$ satisfies the identity $(P_1)$, i.e. $x \circ y \vee (x \circ y) \circ y = 1$.

Indeed, it is not hard to see that any principal filter $[a)$ of $N_5$ satisfies the equality $x^a \vee (x^a)^a = 1$, therefore $N_5$ is a hereditary weakly $(L_1)$-lattice. In view of Theorem 3.3, $(N_5, \wedge, \vee, \circ, 1)$ satisfies $(P_1)$, too.

Let $\mathcal{P}_n$ denote the class of all algebras $(L, \wedge, \vee, \circ, 1)$ corresponding to hereditary weakly $(L_n)$-lattices. Because any $\mathcal{P}_n$ is a subclass of $\mathcal{P}$ determined by the identity $(P_n)$, any $\mathcal{P}_n$ is a subvariety of $\mathcal{P}$. Since in the variety $\mathcal{B}$ of Brouwerian algebras the identity $(P_n)$ is the same as $(L_n)$', the subvarieties $\mathcal{B}_n = \mathcal{B} \cap \mathcal{P}_n$ of $\mathcal{B}$ consist of algebras $(L, \wedge, \vee, *, 1)$ corresponding to relative $(L_n)$-lattices. As in view of [12] relative $(L_n)$-lattices form a proper subclass of the class of relative $(L_{n+1})$-lattices, we have $\mathcal{B}_n \subset \mathcal{B}_{n+1}$. Let $\mathcal{B}_{-1}$ and $\mathcal{B}_0$ be the class of all trivial Brouwerian algebras and the subclass of $\mathcal{B}$ determined by the identity $x \vee y \vee (x * y) = 1$, respectively. Clearly, $\mathbb{L} = (L, \wedge, \vee, *, 1) \in \mathcal{B}_0 \Leftrightarrow$ any $[y, 1]$ is complemented $\Leftrightarrow$ every interval of $L$ is a Boolean lattice. Hence $\mathbb{L} \in \mathcal{B}_0$ if and only if the dual of $L$ is a *generalized Boolean lattice* (see e.g. [9] or [1]).

**Proposition 3.5.** *The variety $\mathcal{P}$ contains an infinite chain of proper subvarieties $\mathcal{B}_{-1} \subset \mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{P}_1 \subset \ldots \subset \mathcal{P}_n \subset \ldots$ and $\mathcal{B}_0$ is the single minimal subvariety of $\mathcal{P}$.*

Proof. The inclusions $\mathcal{B}_{-1} \subset \mathcal{B}_0 \subset \mathcal{B}_1$, $\mathcal{B}_1 \subseteqq \mathcal{P}_1$, $\mathcal{P}_n \subseteqq \mathcal{P}$ are obvious. As $(N_5, \wedge, \vee, \circ, 1)$ is in $\mathcal{P}_1 \setminus \mathcal{B}$ (see Example 3.4) $\mathcal{B}_1 \subset \mathcal{P}_1$ is also clear. Since for any $x_1, \ldots, x_n, x_{n+1}, y \in L$ by Lemma 3.2 we get

$$\begin{aligned}(x_1 \wedge \ldots \wedge x_n \wedge x_{n+1}) \circ y \vee (x_1 \circ y \wedge \ldots \wedge x_n \wedge x_{n+1}) \circ y \vee \ldots \\ \vee (x_1 \wedge \ldots \wedge x_n \circ y \wedge x_{n+1}) \circ y \vee (x_1 \wedge \ldots \wedge x_n \wedge x_{n+1} \circ y) \circ y \\ \geqq (x_1 \wedge \ldots \wedge x_n) \circ y \vee (x_1 \circ y \wedge \ldots \wedge x_n) \circ y \vee \ldots \\ \vee (x_1 \wedge \ldots \wedge x_n \circ y) \circ y,\end{aligned}$$

the identity $(P_n)$ implies $(P_{n+1})$ and this proves $\mathcal{P}_n \subseteqq \mathcal{P}_{n+1}$. Now $\mathcal{B} \cap \mathcal{P}_n \subset \mathcal{B} \cap \mathcal{P}_{n+1}$ implies $\mathcal{P}_n \neq \mathcal{P}_{n+1}$ and $\mathcal{P}_n \neq \mathcal{P}$.

It is known that $\mathcal{B}_0$ is a minimal variety generated by $\mathbb{B}_2$, the Brouwerian algebra defined on the chain with two elements (see e.g. [2]). Let $\mathcal{M}$ be a minimal subvariety of $\mathcal{P}$ and $\mathbb{A} = (A, \wedge, \vee, \circ, 1)$ a nontrivial algebra in $\mathcal{M}$. Then there exists an $a \in A \setminus \{1\}$. As $(\{a, 1\}, \wedge, \vee, \circ, 1)$ is subalgebra of $\mathbb{A}$ isomorphic to $\mathbb{B}_2$, we get $\mathcal{B}_0 = \mathcal{M}$. □

## 4. Application to finite sublattices of a free lattice

In this section we show that any finite sublattice of a free lattice satisfies the equations $(L_4)$ and $(P_4)$.

Since any finite sublattice $L$ of a free lattice is an algebraic $\wedge$-semidistributive lattice (see e.g. Theorem 2.4 in [6]), according to Proposition 2.2 any principal filter of such a lattice $L$ is a pseudocomplemented lattice. Moreover, we prove:

**Theorem 4.1.** *Any finite sublattice of a free lattice is a hereditary weakly* $(L_4)$*-lattice.*

PROOF. As any principal filter $[a)$ of a finite sublattice of a free lattice $F$ is also a finite sublattice of $F$, it is enough to prove that each finite sublattice $L$ of $F$ satisfies the identity $(L_4)$.

In view of [13, Corollary 3.9], a lattice $L$ satisfying the descending chain condition is a weakly $(L_n)$-lattice whenever under each join-irreducible element of $L$ are at most $n$ atoms. In [7] is proved that any finite sublattice $L$ of a free lattice has a *breadth* at most 4 (see also [6, Corollary 5.5]), i.e. for any $n \geqq 4$ and any finite set $\{a_1, a_2, \dots, a_n\}$ there exist $a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4} \in \{a_1, a_2, \dots, a_n\}$ such that $\bigvee_{i=1}^n a_i = a_{i_1} \vee a_{i_2} \vee a_{i_3} \vee a_{i_4}$.

Now, let $L$ be a finite sublattice of a free lattice and $p$ a join-irreducible element of $L$. We show that under $p$ are at most 4 atoms.

Indeed, let us denote by $a_1, a_2, \dots, a_n$ the atoms of $L$ which are under $p$. Then there are at most four atoms $a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4} \in [0, p]$ such that $\bigvee_{i=1}^n a_i = a_{i_1} \vee a_{i_2} \vee a_{i_3} \vee a_{i_4}$. If $n > 4$, then there exist an atom $a_{i_0}$, $i_0 \in \{1, \dots, n\}$ such that $a_{i_0} \notin \{a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}\}$. As $L$ is $\wedge$-semidistributive, the relations $a_{i_1} \wedge a_{i_0} = 0$, $a_{i_2} \wedge a_{i_0} = 0$, $a_{i_3} \wedge a_{i_0} = 0$ and $a_{i_4} \wedge a_{i_0} = 0$ imply $a_{i_0} = (a_{i_1} \vee a_{i_2} \vee a_{i_3} \vee a_{i_4}) \wedge a_{i_0} = 0$, – a contradiction.

Hence $L$ satisfies the identity $(L_4)$. □

Using the above result and applying Theorem 3.3 we obtain:

**Corollary 4.2.** *If* $L$ *is a finite sublattice of a free lattice, then* $L$ *is sectionally pseudocomplemented and the algebra* $(L, \wedge, \vee, \circ, 1)$ *satisfies the equation* $(P_4)$.

## 5. On the congruence properties of the variety $\mathcal{P}$

In this section we shall study the congruence properties of algebras $\mathbb{L} = (L, \wedge, \vee, \circ, 1)$ from the variety $\mathcal{P}$.

Of course, the variety $\mathcal{P}$ is congruence distributive because its reduct to the signature $\{\wedge, \vee\}$ is a class of lattices. Moreover, $\mathcal{P}$ is also congruence permutable. Indeed, one can deduce that a Mal'cev term of $\mathcal{P}$ can be e.g.

$$p(x, y, z) = (y \circ x) \wedge (x \vee z) \wedge (y \circ z).$$

We recall that a variety is *arithmetical* if it is congruence distributive and congruence permutable at the same time. Thus we have:

**Theorem 5.1.** *The variety $\mathcal{P}$ is arithmetical.*

Let $\Theta \in \operatorname{Con} \mathbb{L}$. The set $[1]_\Theta = \{x \in L \mid (1, x) \in \Theta\}$ is called the *kernel of* $\Theta$. We say that the set $K \subseteqq L$ is a *congruence kernel* if $K = [1]_\Theta$ for some $\Theta \in \operatorname{Con} \mathbb{L}$.

Recall that $\mathbb{L}$ is 1-*regular* if $[1]_\Theta = [1]_\Phi$ implies $\Theta = \Phi$ for each $\Theta, \Phi \in \operatorname{Con} \mathbb{L}$. The following result was given by B. Csákány in [5]:

**Proposition 5.2.** *A variety $\mathcal{V}$ with the constant 1 is 1-regular if and only if there exist $n \in \mathbb{N}$ and binary terms $b_1(x, y), \dots, b_n(x, y)$ such that $\mathcal{V}$ satisfies the equivalence*

$$b_1(x, y) = \dots = b_n(x, y) = 1 \iff x = y.$$

By using this proposition we can prove:

**Theorem 5.3.** *The variety $\mathcal{P}$ is* 1*-regular.*

Proof. Take $n = 2$ and $b_1(x, y) = x \circ y$, $b_2(x, y) = y \circ x$. Of course, $b_1(x, x) = b_2(x, x) = 1$. Conversely, suppose $b_1(x, y) = b_2(x, y) = 1$.

Then $x \circ y = 1$ implies $(x \vee y)^y = 1$, i.e. $x \vee y = y$ and $y \circ x = 1$ implies $(x \vee y)^x = 1$, i.e. $x \vee y = x$, thus we get $x = y$. □

*Remark 5.4.* (i) We can get also the Pixley term for arithmecity of $\mathcal{P}$, which is $t(x, y, z) = [(x \circ y) \circ z] \wedge [(z \circ y) \circ x] \wedge (x \vee z)$.

(ii) Let us note, as shown in [3], that a variety $\mathcal{V}$ is 1-regular and permutable if and only if there exist $n \in \mathbb{N}$, binary terms $s_1(x, y), \dots, s_n(x, y)$ and a $(2 + n)$-ary term $q$ such that $\mathcal{V}$ satisfies the identities

$$s_i(x, x) = 1, \quad \text{for } i = 1, \dots, n$$

$$x = q(x, y, s_1(x, y), \dots, s_n(x, y)),$$

$$y = q(x, y, 1, \dots, 1).$$

To verify this Mal'cev condition, one can take $n = 2$, $s_1(x, y) = x \circ y$, $s_2(x, y) = y \circ x$ and

$$q(x, y, z, v) = (z \circ y) \wedge [(v \circ (y \circ x)) \circ x] \wedge (x \vee y).$$

This term $q$ will be also important in the proof of the Theorem 5.7.

Since the variety $\mathcal{P}$ is 1-regular, every congruence $\Theta \in \operatorname{Con} \mathbb{L}$ for $\mathbb{L} \in \mathcal{P}$ is determined by its kernel $[1]_\Theta$. Hence, our task is to determine the congruence kernels and get an explicit description of a congruence determined by a given kernel.

Let $K \subseteqq L$ and let $t(x_1, \dots, x_n, y_1, \dots, y_k)$ be a term of $\mathbb{L} = (L, \wedge, \vee, \circ, 1)$ in two sorts of variables. We say that $K$ is *$y$-closed with respect to $t$* if $t(a_1, \dots, a_n, b_1, \dots, b_k) \in K$ whenever $b_1, \dots, b_k \in K$ and for each $a_1, \dots, a_n \in L$.

For the sake of brevity, we introduce the notations:

$$Q_1 = q(x_1, x_2, y_1, y_2) = (y_1 \circ x_2) \wedge [(y_2 \circ (x_2 \circ x_1)) \circ x_1] \wedge (x_1 \vee x_2),$$

$$Q_2 = q(x_3, x_4, y_3, y_4) = (y_3 \circ x_4) \wedge [(y_4 \circ (x_4 \circ x_3)) \circ x_3] \wedge (x_3 \vee x_4).$$

Further, define the following terms in two sorts of variables:

$$t_1(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (Q_1 \circ Q_2) \circ (x_2 \circ x_4),$$

$$t_2(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (x_2 \circ x_4) \circ (Q_1 \circ Q_2),$$

$$t_3(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (Q_1 \wedge Q_2) \circ (x_2 \wedge x_4),$$

$$t_4(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (x_2 \wedge x_4) \circ (Q_1 \wedge Q_2),$$

$$t_5(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (Q_1 \vee Q_2) \circ (x_2 \vee x_4),$$

$$t_6(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (x_2 \vee x_4) \circ (Q_1 \vee Q_2).$$

**Lemma 5.5.** (i) *Let $K = [1]_\Theta$ for some $\Theta \in \operatorname{Con} \mathbb{L}$ and $t(x_1, \dots, x_n, y_1, \dots, y_k)$ be a term of $\mathbb{L}$ such that $t(x_1, \dots, x_n, 1, \dots, 1)=1$. If $y_1, \dots, y_k \in K$ then $t(x_1, \dots, x_n, y_1, \dots, y_k) \in K$.*

(ii) *For the terms* $t_1, \dots, t_6$ *defined above, we have*

$$t_i(x_1, x_2, x_3, x_4, 1, 1, 1, 1) = 1.$$

Proof. The proof of (i) is elementary and hence omitted. (ii) is an easy consequence of $q(x_1, x_2, 1, 1) = x_2$ and $q(x_3, x_4, 1, 1) = x_4$. □

**Lemma 5.6.** *Let* $K$ *be a subset of* $L$ *with* $1 \in K$*. Define the relation* $\Phi_K$ *on* $L$ *by*

$$(x, y) \in \Phi_K \iff x \circ y \in K \quad \text{and} \quad y \circ x \in K. \tag{$*$}$$

*Then* $K = [1]_{\Phi_K}$.

Proof. If $a \in K$, then $1 \circ a = a \in K$ and $a \circ 1 = 1 \in K$, thus by $(*)$ $(1, a) \in \Phi_K$ and so $a \in [1]_{\Phi_K}$. Conversely, if $a \in [1]_{\Phi_K}$, then $(*)$ gives $a = 1 \circ a \in K$. Thus $K = [1]_{\Phi_K}$. □

A filter $F$ of a lattice $L$ is called *standard* if it is a standard element in the lattice $\mathcal{F}(L)$ of all filters of $L$, i.e. if the equality $[a) \wedge (F \vee [b)) = ([a) \wedge F) \wedge [a \vee b)$ holds in $\mathcal{F}(L)$.

**Theorem 5.7.** *Let* $\mathbb{L} = (L, \wedge, \vee, \circ, 1) \in \mathcal{P}$ *and* $K \subseteqq L$ *with* $1 \in K$*. Then the following assertions are equivalent:*

(i) *$K$ is a congruence kernel.*

(ii) *$K$ is $y$-closed with respect to the terms $t_1, \dots, t_6$.*

(iii) *The relation $\Phi_K$ defined by $(*)$ is a congruence of $\mathbb{L}$.*

(iv) *$K$ is a standard filter of $L$.*

Proof. (i) $\implies$ (ii): Assume that $K = [1]_\Theta$ for some $\Theta \in \operatorname{Con} \mathbb{L}$. Then $1 \in K$ and, by Lemma 5.5, $K$ is $y$-closed with respect to $t_1, \dots, t_6$.

(ii) $\implies$ (iii): Obviously, $\Phi_K$ is reflexive. Suppose $(a, b) \in \Phi_K$ and $(c, d) \in \Phi_K$ for some $a, b, c, d \in L$. Then $a \circ b, b \circ a, c \circ d, d \circ c \in K$. Since in view of Remark 5.4(ii) we have

$$a = q(a, b, s_1(a, b), s_2(a, b)) = q(a, b, a \circ b, b \circ a)$$

and

$$c = q(c, d, s_1(c, d), s_2(c, d)) = q(c, d, c \circ d, d \circ c),$$

and since $K$ is $y$-closed with respect to $t_1$, applying the term $t_1$, we get $(a\circ c)\circ(b\circ d) = (q(a,b,a\circ b,b\circ a)\circ q(c,d,c\circ d,d\circ c))\circ(b\circ d) = t_1(a,b,c,d,a\circ b,b\circ a,c\circ d,d\circ c) \in K$.

Analogously we can show $(b\circ d)\circ(a\circ c) \in K$ applying $t_2$ instead of $t_1$. Hence, by $(*)$ we have also $(a\circ c, b\circ d) \in \Phi_K$.

Substituting $t_3$ and $t_4$ (instead of $t_1$ and $t_2$) in the above argument, we get $(a\wedge c, b\wedge d) \in \Phi_K$ and for $t_5, t_6$ we get $(a\vee c, b\vee d) \in \Phi_K$.

Together, $\Phi_K$ is a reflexive relation on $L$ having the substitution property with respect to all operations of $\mathbb{L}$. As $\mathbb{L}$ belongs to a Mal'cev variety, by the theorem of Werner [15] we obtain $\Phi_K \in \operatorname{Con}\mathbb{L}$.

(iii) $\Longrightarrow$ (i): Since by Lemma 5.6 we have $K = [1]_{\Phi_K}$ and since by assumption $\Phi_K \in \operatorname{Con}\mathbb{L}$, $K$ is a congruence kernel.

(i) $\Longrightarrow$ (iv): Assume that $\Theta$ is a congruence of $\mathbb{L}$ such that $K = [1]_\Theta$. Since $\Theta$ is also a congruence of the lattice $(L,\wedge,\vee,1)$, it is clear that $[1]_\Theta$ is a lattice filter. Suppose that $(x,y) \in \Theta$. Then $(x\vee y, x\wedge y) \in \Theta$, as $\Theta$ is a lattice congruence. Hence $(x\vee y)\circ(x\wedge y) \in K$, because $\Theta$ is a congruence on $\mathbb{L}$. Since

$$x\wedge y = (x\vee y)\wedge[(x\vee y)\circ(x\wedge y)],$$

$K$ is a standard filter by [8, Theorem III.5].

(iv) $\Longrightarrow$ (i): Assume that $K$ is a standard filter of $L$ and take $\Theta = \Theta[K]$ the smallest congruence on $L$ generated by $K$. This exists by [8, Theorem III.5] and it is easy to check that $K = [1]_\Theta$. We have only to show that $\Theta$ is compatible with the binary operation $\circ$. It is enough to show that the factor-lattice $L/\Theta$ is sectionally pseudocomplemented. More precisely, we claim that $[a]_\Theta \circ [b]_\Theta = [a\circ b]_\Theta$ for any $a, b \in L$. Really,

$$([a]_\Theta \vee [b]_\Theta) \wedge [a\circ b]_\Theta = [a\vee b]_\Theta \wedge [a\circ b]_\Theta = [(a\vee b)\wedge(a\circ b)]_\Theta = [b]_\Theta$$

as $\Theta$ is a lattice congruence.

Now, take $[x]_\Theta \geqq [b]_\Theta$ in $L/\Theta$ and suppose that

$$([a]_\Theta \vee [b]_\Theta) \wedge [x]_\Theta = [(a\vee b)\wedge x]_\Theta = [b]_\Theta.$$

Without loss of generality we can assume $x \geqq b$ in $L$. Then $(a\vee b)\wedge x \geqq b$ and in view of [8, Theorem III.5] there exists a $g \in K$ such that $(a\vee b)\wedge x\wedge g = b$ holds in $L$. It follows that $x\wedge g \leqq a\circ b$, as $L$ is sectionally pseudocomplemented. As $g \in K$, we have $[x\wedge g]_\Theta = [x]_\Theta$.

Finally, we obtain $[x]_\Theta \leqq [a \circ b]_\Theta$ and hence $[a \circ b]_\Theta = [a]_\Theta \circ [b]_\Theta$, as claimed. $\square$

**Corollary 5.8.** *For any* $\Theta \in \operatorname{Con} \mathbb{L}$ *we have* $\Theta = \Phi_{[1]_\Theta}$.

Proof. Let $\Theta \in \operatorname{Con} \mathbb{L}$ and take $K = [1]_\Theta$. Then by Theorem 5.7 we have $\Phi_K \in \operatorname{Con} \mathbb{L}$ and Lemma 5.6 gives $[1]_\Theta = [1]_{\Phi_K}$. As $\mathbb{L}$ is an algebra of a congruence 1-regular variety, we get $\Theta = \Phi_K$, i.e. $\Theta = \Phi_{[1]_\Theta}$. $\square$

## Problems

1) Characterize the subdirectly irreducible algebras in the varieties $\mathcal{P}_n$, $n \in \mathbb{N}$.
2) Characterize the lattice of subvarieties of $\mathcal{P}$.

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