Publ. Math. Debrecen
42 / 3–4 (1993), 409–415

On the exponent of the group of normalized units of a modular group algebras

By A. BOVDI (Debrecen) and P. LAKATOS (Debrecen)

Dedicated to Professor Lajos Tamássy on his 70th birthday

Let G be a finite p-group and K a finite field of characteristic p. The group of units of KG is denoted by $U(KG)$. It is easy to show that $U(KG) = U(K) \times V(KG)$ where

$$V(KG) = \left\{ \sum_{g \in G} \alpha_g g \in KG \mid \sum_{g \in G} \alpha_g = 1, \alpha_g \in K \right\}$$

is the group of normalized units. $V(KG)$ is a p-group and $|V(KG)| = p^{r(|G|-1)}$ where $|K| = p^r$. Clearly $V(KG)$ is a normal Sylow p-subgroup in $U(KG)$.

In general the problem of determining the exponent of $V(KG)$ is open, the first partical result was obtained by Z. Patay [4] and A. Shalev [5]. It is an interesting and important problem.

Since G is embedded in $V(KG)$, we obviously have

$$\exp(V(KG)) \geq \exp(G),$$

but usually the exponent of $V(KG)$ is much larger. Indeed, by the result of Coleman and Passman [4] if G is non-abelian and $p \neq 2$ then the wreath product $C_p wr C_p$ is involved in $V(KG)$, and we get

$$\exp(V(KG)) \geq p^2.$$

Moreover, it turns out that for every $p \neq 2$ there exists a sequence $\{G_m\}_{m \geq 1}$ of finite groups of exponent p, such that $\exp(V(KG_m)) \to \infty$

Research supported by the Hungarian National Foundatin for Scientific Research No. 1903 and 1654.

[5]. For example, we may choose G_m as the free nilpotent group of class 2 and exponent p on m generators. This is a consequence of the theorem of Passman [2] on polynomial identities in group rings of characteristic p. Therefore one cannot expect general inequalities of the form $\exp(V(KG)) \leq f(\exp(G))$ for a fixed function $f : N \to N$. On the other hand, $\exp(V(KG))$ can be small for arbitrary large G. Thus, for example, if G is abelian, then $\exp(V(KG)) = \exp(G)$. Aner Shalev [5] proved that if $p \geq 7$ and $\exp(G)^3 > |G|$, then G and $V(KG)$ have the same exponent. Our main problem is identify situations when $\exp(V(KG))$ is finite and close to $\exp(G)$.

If $1 + A(KG) = V(KG)$ and $V(KG)$ has a finite exponent then by Zelmanov's theorem and by theorem 2.12 [1] G is a locally finite p-group and K is a field of characteristic p.

We proved the following results:

Theorem 1. *Let G be a locally finite p-group. Then $V(KG)$ has finite exponent if and only if G has finite exponent and there exists a normal subgroup L of finite index in G such that the commutator subgroup of L is finite.*

Proof. Let p^n be the exponent of $V(KG)$. Clearly $V(KG) = 1 + A(KG)$ and for every $x \in KG$ there exists such $\alpha \in K$ that $\alpha + x \in V(KG)$. Then the Lie product $((\alpha + x)^{p^n}, y)$ coincides with (x^{p^n}, y) and KG satisfies the polinomial identity $(x^{p^n}, y) = 0$, where $y \in KG$. By Passman's theorem [2] we know the structure of the group G.

Now suppose that G has a normal subgroup L such that $|G/L| = p^m$ and the commutator subgroup L' has order p^t. Let $I(L')$ be an ideal of KG generated by the elements $g - 1$ ($g \in L'$). It is well-known that $I(L')$ nilpotent and

$$V(KG)/(I(L') + 1) \cong V(K(G/L')).$$

Since $1 + I(L')$ is a subgroup of finite exponent without loss of generality we can assume that L is abelian. Let $g_1, \dots , g_{p^m}$ be representatives of the distinct cosets of G modules L. If $x \in V(KG)$, then there exist elements x_i in KL such that

$$x = x_1g_1 + x_2g_2 + \cdots + x_{p^m}g_{p^m}.$$

Every element x_i has finite G-orbit and the order of each orbit is less than $|G : L|$. If $x_i = \sum_{g\in L} \alpha^i_g g$ and $\chi(x_i) = \sum_{g\in L} \alpha^i_g$, then $(x_i - \chi(x_i))^{p^k} = 0$ where p^k is the exponent of L. Since

$$x = 1 + (x_1 - \chi(x_1))g_1 + \cdots + (x_{p^m} - \chi(x_{p^m})g_{p^m},$$

we have that $x^{p^{k+m}} = 1$ which proves the theorem.

Lemma 1. *Let G be such group that*

$$[a^{p^n}, b] = [a, b]^{p^n} c^{p^n} \tag{1}$$

for all $a, b \in G$, where c is in the commutator group of the group generated by a and b. If $w \in I(G')^{p^{n-1}}$ and $x_i \in \{w, g^{p^n}\ (g \in G)\}$ then the Lie product

$$(x_1 x_2 \cdots x_k, x_{k+1} \cdots x_p) \in I(G')^{p^n}. \tag{2}$$

Proof. The following identities hold in KG

$$(uv, w) = (u, v)v + u(v, w), \tag{3}$$

where $u, v, w \in KG$. If $x_i = w$ for all i than the statment of lemma is trivial.

Suppose that the first $x_j = g^{p^n}$ and $x_{j-1} = \cdots = x_1 = w$. Applying (3), we conclude that

$$\begin{aligned}(x_1 \cdots x_{j-1} g^{p^n} \cdots x_k, x_{k+1} \cdots x_{p^n}) &= (w^{j-1}, x_{k+1} \cdots x_{p^n}) g^{p^n} x_{j+1} \cdots x_k + \\ &+ w^{j-1}(g^{p^n} x_{j+1} \cdots x_k, x_{k+1} \cdots x_{p^n}) = \\ = (w^{j-1}, x_{k+1} \cdots x_p) g^{p^n} x_{j+1} \cdots x_k &+ w^{j-1}(g^{p^n}, x_{k+1} \cdots x_{p^n}) x_{j+1} \cdots x_k + \\ &+ w^{j-1} g^{p^n} (x_{j+1} \cdots x_k, x_{k+1} \cdots x_{p^n}).\end{aligned}$$

Let

$$x_{j+1} \cdots x_{p^n} = \sum_{h \in G} \alpha_h h.$$

Then

$$(g^{p^n}, x_{k+1} \cdots x_p) = \sum_{h \in G} \alpha_h (g^{p^n} h - h g^{p^n}) =$$

$$= \sum_{h \in G} \alpha_h g^{p^n} h (1 - [g^{p^n}, h]) = \sum \alpha_h g^{p^n} h (1 - [g, h]^{p^n} c^{p^n}) \in I(G')^{p^n}.$$

If $x_k = w$ $(k = j+1, \cdots, p^n)$, then the proof is complete; otherwise the argument may be repeated until we see that all relevant commutator are in $I(G')^{p^n}$.

Theorem 2. *Let G be a finite p-group with cyclic commutator subgroup G'. If $\exp(G) = \exp(G')$ then $\exp(V(KG)) = p \cdot \exp(G)$, and if $\exp(G) \neq \exp(G')$ then $\exp(V(KG)) = \exp(G)$.*

Proof. Let $G' = \langle c \rangle$ be the cyclic commutator subgroup of order p^n. The lower central series of G will be denoted by

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_s(G).$$

It is well-known the following Hall's collection formula [6]

$$[a^{p^{n-1}}, b] \equiv [a,b]^{p^{n-1}} \operatorname{mod} (\gamma_2(G))^{p^{n-1}} \prod_{1 \le r < n} (\gamma_{p^r}(H))^{p^{n-1-r}}, \tag{4}$$

where H is the subgroup generated by elements $a, [a,b]$. Because $\gamma_2(G) = \langle c \rangle$ and G is nilpotent this formula implies that

$$[a^{p^{n-1}}, b] = [a,b]^{p^{n-1}} c^{kp^{n-1}} \tag{5}$$

for all $a, b \in G$ and k depends on a, b. Let

$$x = \sum_{g \in G} \alpha_g g \in V(KG)$$

and $L(KG) = [KG, KG]$ is the K-modul generated by Lie product (u, v) for any $u, v \in KG$. Then by Proposition 3.1 [1]

$$x^p = \sum_{g \in G} \alpha_g^p g^p + w \quad (w \in L(KG)).$$

It is clear that $L(KG)$ is contained in ideal $I(G')$ of KG generated by all $g - 1$ $(g \in G')$.

Let us prove by induction of on the order p^n of commutator subgroup of G that

$$x^{p^n} \equiv \sum_{g \in G} \alpha_g^{p^n} g^{p^n} (\operatorname{mod} I(G')^{p^{n-1}}) \tag{6}$$

In the case $n = 1$ it is true by (4). Let H be a subgroup of order p of G' and $I(H)$ is an ideal of KG generated by $h - 1$ $(h \in H)$. Then $KG/I(H) \cong KG/H$, the commutator subgroup of G/H has order p^{n-1} and $I(H) \subseteq I(G')^{p^{n-1}}$. Applying the induction hypothesis, we deduce

$$x^{p^{n-1}} + I(H) \equiv \sum_{g \in G} \alpha_g^{p^{n-1}} g^{p^{n-1}} + I(H) (\operatorname{mod} I(G')^{p^{n-2}}/I(H)).$$

Because there exists element $y \in I(G')^{p^{n-2}}$ such that

$$x^{p^{n-1}} = \sum_{g \in G} \alpha_g^{p^{n-1}} g^{p^{n-1}} + y,$$

then

$$x^{p^n} = \sum_{g \in G} \alpha_g^{p^n} g^p + y^p + \sum u_1 u_2 \cdots u_p, \quad u_i \in \{g^{p^{n-1}}, y\},$$

where the sum is taken over all such products $u_1u_2\cdots u_p$, that not all u_i equal to g^{p^n} or y, and the sum contains the cyclic permutation of each word $u_1u_2\cdots u_p$. Because KG has characteristic p and

$$u_iu_{i+1}\cdots u_pu_1u_2\cdots u_{i-1}-u_1u_2\cdots u_p=(u_iu_{i+1}\cdots u_p,u_1u_2\cdots u_{i-1}),$$

then $\sum u_1u_2\cdots u_p$ may be represent as Lie product

$$v=(u_{i_1}u_{i_2}\cdots u_{i_k},u_{i_{k+1}}\cdots u_{i_p}).$$

Let for some Lemma 1 we conclude that $v\in I(G')^{p^n}$. Therefore (6) is true and this we can use in determination of the exponent of $V(KG)$.

Obviously, the element g^{p^n} is in the centre for all $g\in G$ and $I(G')^{p^n}=0$. If $\exp(G)>p^n$ then it follows from (3) that $\exp(G)=\exp(V(KG))$.

Assume that $\exp(G)=\exp(G')=p^n$. By virtue of (6) $\exp(V(KG))\leq$ $\leq p^{n+1}$. There exist such $a,b\in G$ that $c=[a,b]$ and c has order p^n.

We now claim the element $x=1+b^{-1}(a-1)$ has order p^{n+1}.

Suppose that $x^{p^n}=1$. Then $b^{-i}ab^i=ac^{k_i}$ and $[b^{-1}(a-1)]^{p^n}=0$ from which it follows that

$$(ac-1)(ac^{k_2}-1)\cdots(ac^{k_{p^n-1}}-1)(a-1)=0.$$

By proposition 2.7 [1]

(7) $(ac-1)(ac^{k_2}-1)\cdots(ac^{k_{p^n}-1}-1)=(1+a+\cdots+a^{p^n-1})z\ (z\in K\langle c\rangle)$.

It easily verified from (5) that $\langle c\rangle\bigcap\langle a\rangle=1$ since $\langle c\rangle$ is a normal subgroup of $\langle c,a\rangle$ and comparising coefficients of a and 1 in (7) we get

$$c=z,\quad -1=z,$$

which is impossible. Therefore we conclude $\exp(V(KG))=p^{n+1}$.

Theorem 3. *Let G be a finite p-group of nilpotency class two or G is a finite p-regular group. Let $t(G')$ denotes the nilpotency class of the augmentation ideal $A(KG')$ and k is the least integer such that $t(G')\leq p^k$. Then*

1. *if $p^k<\exp(G)$, then $\exp(V(KG))=\exp(G)$;*
2. *if $p^k\geq\exp(G)$, then $\exp(V(KG))\leq p^{k+1}$.*

Proof. If G has the nilpotency class two then (1) is valid and the conditions the Lemma 1 is satisfied.

Immediate consequence of the definition of p-regular group we become

$$[x^{p^{n-1}},y]=[x,y]^{p^{n-1}}c^{p^{n-1}},$$

where c is in the commutator subgroup. Then the argument of proof of the theorem 2 may be repeated and we get the statement of this theorem.

Theorem 4. *Let C be the center of a 2-group G which contains an abelian subgroup A of index two and K be a field of two elements. Then*

1. *if* $\exp(A/C) < \exp(A)$ *then* $\exp(V(KG)) = \exp(G)$,
2. *if* $\exp(A/C) = \exp(A) = \exp(G)$ *then* $\exp(V(KG)) = 2 \cdot \exp(G)$,
3. *if* $\exp(A/C) = \exp(A) < \exp(G)$ *then* $\exp(V(KG)) = \exp(G)$.

Proof. Clearly there exists such $b \in G$ that $G = \langle A, b\rangle$ and $b^2 \in A$. Then every element x of $V(KG)$ has a unique representation in the form $x = x_1 + x_2 b$, where $x_1, x_2 \in KA$. It is easy to see that the map defined by $u \to \bar{u} = b^{-1}ub$ ($u \in KA$) is an involution on KG and

$$x^2 = x_1^2 + x_2\bar{x_2}b^2 + x_2(x_1 + \bar{x_1})b$$

The reader can readily verify by induction that

$$x^{2^n} = x_1^{2^n} + (x_2\bar{x_2})^{2^{n-1}}b^{2^n} + \sum_{i=1}^{n-1}(x_2\bar{x_2})^{2^{i-1}}(x_1+\bar{x_1})^{2^n-2^i}b^{2^i} + \\ + x_2(x_1+\bar{x_1})^{2^n-1}b\,. \tag{8}$$

Let $\exp(A) = 2^m$, $\exp(A/C) = 2^t$ and $x_1 = \sum_{a\in A}\alpha_a a$. Then $x_1+\bar{x_1}= \sum_{a\in A}\alpha_a(a+\bar{a})$ and $(a+\bar{a})^{2^t} = 0$, because $(x_1+\bar{x_1})^{2^t} = 0$ and

$$(x_2\bar{x_2})^{2^t} = x_2^{2^{t+1}}.$$

Suppose that $t < m$. It is clear that $2^m - 2^i \geq 2^{m-1}$ for every $i \leq m-1$. By (8) we have

$$x^{2^m} = \chi(x_1)^{2^m} + \chi(x_2)^{2^m}b^{2^m},$$

where $\chi(x_1) = \sum_{a\in A}\alpha_a$. Suppose that $t = m$. Then

$$x^{2^m} = \chi(x_1)^{2^m} + (x_2\bar{x_2})^{2^{m-1}}b^{2^m}$$

and we conclude

$$x^{2^{m+1}} = \chi(x_1)^{2^{m+1}} + \chi(x_2)^{2^{m+1}}b^{2^{m+1}}\,. \tag{9}$$

Because $x \in V(KG)$ we have $\chi(x_1) + \chi(x_2) = 1$. If $\exp(A) < \exp(V)$, then (9) implies that $\exp(V(KG)) = \exp(G)$.

Assume that $\exp(A/C) = \exp(A) = \exp(G)$. Clearly there exist cyclic subgroups $\langle a_1\rangle, \langle a_2\rangle$ of order 2^m in A such that $\langle ba_1b^{-1}\rangle \cap \langle a_2\rangle = 1$. Then one immediately verifies that $x = 1 + (a_1 + a_2)b$ has order 2^{m+1}. Thus $\exp(V(KG)) = 2\exp(G)$, which proves the theorem.

References

[1] A. Bovdi, Group rings, *Uchebo-metodicheskij kabinet Visheho obrazovanija*, *Kiew*, 1988. (in Russian)

[2] D. S. Passman, Group rings satisfying a polynomial identity, *J. Algebra* **20** (1972), 103–117.

[3] Coleman and D. S. Passman, Units in modular group rings, *Proc Amer Math Soc.* **25** (1970), 170–175.

[4] Z. Patay, On the structure of the center of the unit group of the group ring, Dissertation, *Uzhgorod*, 1975.

[5] A. Shalev, Lie dimension subgroups, Lie nilpotency indices, and the exponent of the group of normalized units, *J. Math. Soc. 2* **43** (1991), 23–31.

[6] B. Huppert and N. Blackburn, Finite groups II, *Springer–Verlag*, *Berlin–New York*, 1967.

A. BOVDI
KOSSUTH LAJOS UNIVERSITY
INSTITUTE OF MATHEMATICS
H–4010 DEBRECEN, PF. 12.
HUNGARY

P. LAKATOS
KOSSUTH LAJOS UNIVERSITY
INSTITUTE OF MATHEMATICS
H–4010 DEBRECEN, PF. 12.
HUNGARY

(Received February 12, 1993)