

A note on spheres in a Euclidean space

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Abstract. For an orientable compact and connected positively curved hypersurface in the Euclidean space R^{n+1} , $n \geq 2$, with scalar curvature S , shape operator A and mean curvature α , it is shown that the inequality

$$\|A\|^2 S \geq \frac{1}{2} \|R\|^2 + \|Q\|^2 + 2n(n-1) \|\nabla\alpha\|^2$$

implies that the hypersurface is a sphere, where $\nabla\alpha$ is the gradient of α , and $\|R\|$, $\|Q\|$ are the lengths of the curvature tensor field R , the Ricci operator Q of the hypersurface respectively.

1. Introduction

The class of positively curved compact hypersurfaces in the Euclidean space R^{n+1} is quite large and therefore it is an interesting question in Geometry to obtain conditions which characterize the spheres in this class. We denote by R and Ric the curvature tensor field and the Ricci curvature tensor field of the hypersurface M of the Euclidean space R^{n+1} . The Ricci operator Q is defined as $\text{Ric}(X, Y) = g(QX, Y)$, $X, Y \in \mathfrak{X}(M)$, where g is the induced metric and $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M . For a local orthonormal frame $\{e_1, \dots, e_n\}$ on M the lengths $\|R\|$ and $\|Q\|$ are respectively given by $\|R\|^2 = \sum_{ijk} \|R(e_i, e_j)e_k\|^2$,

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$\|Q\|^2 = \sum_{ij} g(Qe_i, e_j)^2$. Let A be the shape operator of the hypersurface, α its mean curvature and S be its scalar curvature. An interesting question in Geometry is using the invariants α , S , $\|A\|$, $\|R\|$, $\|Q\|$ of the hypersurface, how to characterize the spheres in R^{n+1} ? For instance, a sphere $S^n(c)$ in R^{n+1} , satisfies the equality

$$\|A\|^2 S = \frac{1}{2} \|R\|^2 + \|Q\|^2 + 2n(n-1) \|\nabla\alpha\|^2$$

$\nabla\alpha$ being the gradient of the mean curvature α . This raises a question, does a compact hypersurface satisfying above equality necessarily a sphere? In this paper we show that the answer is in affirmative for positively curved hypersurfaces, and indeed we prove the following:

Theorem. *Let M be an orientable compact and connected positively curved hypersurface of the Euclidean space R^{n+1} , $n \geq 2$. If the scalar curvature S , the shape operator A , the mean curvature α , the curvature tensor field R and the Ricci operator Q of M satisfy*

$$\|A\|^2 S \geq \frac{1}{2} \|R\|^2 + \|Q\|^2 + 2n(n-1) \|\nabla\alpha\|^2,$$

then α is a constant and $M = S^n(\alpha^2)$.

2. Preliminaries

Let M be an orientable hypersurface of the Euclidean space R^{n+1} . We denote the induced metric on M by g . Let $\bar{\nabla}$ be the Euclidean connection and ∇ be the Riemannian connection on M with respect to the induced metric g . Let N be the unit normal vector field and A be the shape operator. Then the Gauss and Weingarten formulas for the hypersurface are

$$\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \bar{\nabla}_X N = -AX, \quad X, Y \in \mathfrak{X}(M) \quad (2.1)$$

where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M . We also have the following Gauss and Codazzi equations

$$R(X, Y)Z = g(AY, Z)AX - g(AX, Z)AY \quad (2.2)$$

$$(\nabla A)(X, Y) = (\nabla A)(Y, X), \quad X, Y, Z \in \mathfrak{X}(M) \quad (2.3)$$

where R is the curvature tensor field of the hypersurface and $(\nabla A)(X, Y) = \nabla_X AY - A\nabla_X Y$. The mean curvature α of the hypersurface is given by $n\alpha = \sum_i g(Ae_i, e_i)$, where $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M . If $A = \lambda I$ holds for a constant λ , then the hypersurface is said to be totally umbilical. The square of the length of the shape operator A is given by

$$\|A\|^2 = \sum_{ij} g(Ae_i, e_j)^2 = \text{tr}.A^2.$$

From equation (2.2) we get the following expression for the Ricci tensor field

$$\text{Ric}(X, Y) = n\alpha g(AX, Y) - g(AX, AY). \quad (2.4)$$

The scalar curvature S of the hypersurface is given by

$$S = n^2\alpha^2 - \|A\|^2. \quad (2.5)$$

The Ricci operator Q is the symmetric operator $Q : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ defined by $\text{Ric}(X, Y) = g(QX, Y)$, $X, Y \in \mathfrak{X}(M)$. Then from equation (2.4) we get

$$Q = n\alpha A - A^2. \quad (2.6)$$

3. Some lemmas

Let M be a hypersurface of R^{n+1} and $\nabla\alpha$ be the gradient of the mean curvature function α . Then we have

Lemma 3.1. *Let M be an orientable hypersurface of R^{n+1} and $\{e_1, \dots, e_n\}$ be a local orthonormal frame on the hypersurface M . Then*

$$\sum_i (\nabla A)(e_i, e_i) = n\nabla\alpha.$$

The proof is straightforward and follows from the symmetry of A and the equation (2.3).

Lemma 3.2. *Let M be an orientable hypersurface of R^{n+1} . Then the length $\|R\|$ of the curvature tensor field of the hypersurface M is given by*

$$\frac{1}{2} \|R\|^2 = \|A\|^4 - \|A^2\|^2.$$

The proof follows immediately from equation (2.2).

Lemma 3.3. *Let M be an orientable hypersurface of R^{n+1} . Then the length $\|Q\|$ of the Ricci operator Q of the hypersurface M is given by*

$$\|Q\|^2 = n^2 \alpha^2 \|A\|^2 + \|A^2\|^2 - 2n\alpha(\text{tr}.A^3)$$

The proof follows immediately from the equation (2.6).

Lemma 3.4. *Let M be an orientable hypersurface of R^{n+1} , $n \geq 2$. Then*

$$\|\nabla A\|^2 \geq n \|\nabla \alpha\|^2,$$

where $\|\nabla A\|^2 = \sum_{ij} \|(\nabla A)(e_i, e_j)\|^2$ for a local orthonormal frame $\{e_1, \dots, e_n\}$ on M , moreover for a positively curved M if the equality holds then M is totally umbilical.

PROOF. Define an operator $B : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ by $B = A - \alpha I$. Then we have

$$(\nabla B)(X, Y) = (\nabla A)(X, Y) - (X\alpha)Y,$$

which gives

$$\begin{aligned} \|\nabla B\|^2 &= \|\nabla A\|^2 + n \|\nabla \alpha\|^2 - 2 \sum_{ij} g((\nabla A)(e_i, e_j), e_j) g(\nabla \alpha, e_i) \\ &= \|\nabla A\|^2 + n \|\nabla \alpha\|^2 - 2 \sum_j g(\nabla \alpha, (\nabla A)(e_j, e_j)) \\ &= \|\nabla A\|^2 - n \|\nabla \alpha\|^2. \end{aligned}$$

This proves that $\|\nabla A\|^2 \geq n \|\nabla \alpha\|^2$. The equality holds if and only if $\nabla B = 0$. If M is positively curved, then in this case we shall have $B = \lambda I$ for some constant λ (as M is irreducible being positively curved). However $B = A - \alpha I$ gives that $\text{tr}.B = 0$. Hence $\lambda = 0$ and consequently $B = 0$ that is $A = \alpha I$ and thus M is totally umbilical ($\alpha = \text{a constant}$, follows from equation (2.3) and $\dim M \geq 2$). $\square$

Lemma 3.5. *Let M be an orientable compact hypersurface of the Euclidean space R^{n+1} . Then*

$$\int_M \left(\sum_i g(\nabla_{e_i}(\nabla\alpha), Ae_i) \right) dV = -n \int_M \|\nabla\alpha\|^2 dV$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M .

PROOF. Choosing a point wise covariant constant local orthonormal frame $\{e_1, \dots, e_n\}$ on M , we compute

$$\begin{aligned} \operatorname{div}(A(\nabla\alpha)) &= \sum_i e_i g(\nabla\alpha, Ae_i) \\ &= \sum_i g(\nabla_{e_i}(\nabla\alpha), Ae_i) + \sum_i g(\nabla\alpha, (\nabla A)(e_i, e_i)) \\ &= \sum_i g(\nabla_{e_i}(\nabla\alpha), Ae_i) + n \|\nabla\alpha\|^2. \end{aligned}$$

Integrating this equation we get the lemma. $\square$

We define the second covariant derivative $(\nabla^2 A)(X, Y, Z)$ as

$$(\nabla^2 A)(X, Y, Z) = \nabla_X(\nabla A)(Y, Z) - A(\nabla_X Y, Z) - A(Y, \nabla_X Z),$$

then using the Ricci identity we get

$$(\nabla^2 A)(X, Y, Z) - (\nabla^2 A)(Y, X, Z) = R(X, Y)AZ - AR(X, Y)Z. \quad (3.1)$$

4. Proof of the theorem

Let M be an orientable compact and connected hypersurface of the Euclidean space R^{n+1} . Define a function $f : M \rightarrow R$ by $f = \frac{1}{2} \|A\|^2$. Then by a straightforward computation we get the Laplacian Δf of the smooth function f as

$$\Delta f = \|\nabla A\|^2 + \sum_{ij} g((\nabla^2 A)(e_j, e_j, e_i), Ae_i) \quad (4.1)$$

where $\{e_1, \dots, e_n\}$ is local orthonormal frame on M .

Using the equation (2.3), we arrive at

$$g((\nabla^2 A)(e_j, e_j, e_i), Ae_i) = g((\nabla^2 A)(e_j, e_i, e_j), Ae_i).$$

Now using the Ricci identity (3.1) in above equation we get

$$\begin{aligned} g((\nabla^2 A)(e_j, e_j, e_i), Ae_i) &= g((\nabla^2 A)(e_i, e_j, e_j), Ae_i) \\ &\quad + g(R(e_j, e_i)Ae_j, Ae_i) - g(R(e_j, e_i)e_j, A^2e_i). \end{aligned}$$

Thus in light of this equation the equation (4.1) takes the form

$$\begin{aligned} \Delta f &= \|\nabla A\|^2 + \sum_{ij} g((\nabla^2 A)(e_i, e_j, e_j), Ae_i) \\ &\quad + \sum_{ij} [g(R(e_j, e_i)Ae_j, Ae_i) - g(R(e_j, e_i)e_j, A^2e_i)]. \end{aligned} \quad (4.2)$$

Using Lemma 3.1, we get

$$\sum_j (\nabla^2 A)(e_i, e_j, e_j) = n \nabla_{e_i}(\nabla \alpha). \quad (4.3)$$

Now we use equations (2.2) and (2.4) to compute

$$\begin{aligned} \sum_{ij} [g(R(e_j, e_i)Ae_j, Ae_i) - g(R(e_j, e_i)e_j, A^2e_i)] \\ = \|A^2\|^2 - \|A\|^4 - [\|A^2\|^2 - n\alpha(\text{tr}.A^3)] = n\alpha(\text{tr}.A^3) - \|A\|^4. \end{aligned}$$

Using this last equation together with (4.3) in (4.2), we arrive at

$$\Delta f = \|\nabla A\|^2 + n \sum_i g(\nabla_{e_i}(\nabla \alpha), Ae_i) + n\alpha(\text{tr}.A^3) - \|A\|^4.$$

Integrating this equation and using Lemmas 3.2, 3.3 and 3.5 and equation (2.5) we arrive at

$$\begin{aligned} \int_M \left\{ \left[\|\nabla A\|^2 - n \|\nabla \alpha\|^2 \right] + \frac{1}{2} \|A\|^2 S \right. \\ \left. - \frac{1}{2} \left(\|Q\|^2 + \frac{1}{2} \|R\|^2 + 2n(n+1) \|\nabla \alpha\|^2 \right) \right\} dV = 0. \end{aligned}$$

The condition in the statement of the theorem together with Lemma 3.4 and above equation yields

$$\|\nabla A\|^2 = n \|\nabla \alpha\|^2.$$

Since M is positively curved, the above equality again in view of Lemma 3.4 gives that M is totally umbilical hypersurface of R^{n+1} and thus the theorem is proved.

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