

Totally real Einstein submanifolds of CP^n and the spectrum of the Jacobi operator

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Abstract. We consider n -dimensional compact totally real parallel Einstein submanifolds of the complex projective space CP^n and we use invariants determined by the spectrum of the Jacobi operator J to characterize such submanifolds.

1. Introduction

Let M be an n -dimensional compact (connected and smooth) Riemannian manifold without boundary, isometrically immersed in a Riemannian manifold $\bar{M}$. Then a second order elliptic operator J , called the *Jacobi operator*, is associated to the isometric immersion. Such operator is defined on the space of smooth sections of the normal bundle $TM^\perp$ by the formula

$$J = D + \tilde{R} - \tilde{A},$$

where D is the rough Laplacian of the normal connection $\nabla^\perp$ on $TM^\perp$, $\tilde{R}$ and $\tilde{A}$ are linear transformations of $TM^\perp$ defined by means of a partial Ricci tensor of $\bar{M}$ and of the second fundamental form A , respectively. J is also called the *second variation operator* because it appears in the formula which gives the second variation for the area function of a compact minimal

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submanifold (see [15]). Its spectrum, denoted by

$$\text{spec}(M, J) = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots + \uparrow \infty\}$$

is discrete, as a consequence of the compactness of M .

H. DONNELLY [4] and T. HASEGAWA [8], applying GILKEY's results [6] to the asymptotic expansion of the partition function $Z(t)$ associated to $\text{spec}(M, J)$, found spectral invariants and studied spectral geometry for compact minimal submanifolds of the Euclidean sphere and for compact Kaehlerian submanifolds of the complex projective space CP^n . Some results about spectral geometry of Sasakian submanifolds were given in [14]. Moreover, an analogous study was made about spectral geometry determined by the Jacobi operator associated to the energy of a harmonic map in [16] and in [12] for Riemannian foliations. Recently, the inverse spectral problem of the Jacobi operator of a harmonic map has been further investigated in [2], [9], [10], [17].

Besides Kaehlerian submanifolds, another typical class of submanifolds of the complex projective space CP^n is the one of totally real minimal submanifolds. In [1], the author and D. Perrone determined the first three terms of the asymptotic expansion for the partition function associated to the spectrum of the Jacobi operator of an n -dimensional totally real submanifold of CP^n . The corresponding Riemannian spectral invariants have been used to characterize n -dimensional totally real parallel conformally flat submanifolds of CP^n .

In this paper, we use Riemannian invariants determined by $\text{spec}(M, J)$ to characterize n -dimensional totally real parallel Einstein submanifolds of CP^n . In Section 2 we shall make some preliminaries about n -dimensional totally real minimal submanifolds of CP^n . Section 3 is devoted to the description of n -dimensional totally real parallel Einstein submanifolds of CP^n . In Section 4, we shall characterize such submanifolds, for a wide range of dimensions, using some spectral invariants of J .

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2. Preliminaries

Let CP^n denote the complex projective space equipped with the Fubini-Study metric $\bar{g}$ of constant holomorphic sectional curvature $c > 0$. An n -dimensional *totally real submanifold* of CP^n is a Riemannian manifold (M, g) isometrically immersed in CP^n such that $IT_x M$ is orthogonal to $T_x M$ for all $x \in M$, where I denotes the almost complex structure of CP^n . We shall denote by ∇ (respectively, $\bar{\nabla}$) and R (respectively, $\bar{R}$) the Levi-Civita connection and the curvature tensor of M (respectively, CP^n), taken with the sign convention

$$R(X, Y) = \nabla_{[X, Y]} - [\nabla_X, \nabla_Y].$$

Note that this sign convention is the opposite from that used by SIMONS in [15].

The normal connection is defined by

$$\begin{aligned} \nabla^\perp : TM \times TM^\perp &\longrightarrow TM^\perp \\ (X, \xi) &\longmapsto \nabla_X^\perp \xi, \end{aligned}$$

where $\nabla_X^\perp \xi$ denotes the normal component of $\bar{\nabla}_X \xi$. The second fundamental form σ and the Weingarten operator A are respectively defined by

$$\sigma(X, Y) = \bar{\nabla}_X Y - \nabla_X Y, \quad A_\xi X = -\bar{\nabla}_X \xi + \nabla_X^\perp \xi,$$

for all $X, Y \in TM$ and $\xi \in TM^\perp$. Moreover, $\bar{g}(\sigma(X, Y), \xi) = g(A_\xi X, Y)$ and, since M is totally real, $A_{IX} Y = A_{IY} X$, for all $X, Y \in TM$ and $\xi \in TM^\perp$ (see [3]).

Let $R^\perp$ denote the curvature tensor associated to the normal connection $\nabla^\perp$. The curvature tensors R , $\bar{R}$ and $R^\perp$ satisfy the Gauss and the Ricci equations:

$$\begin{aligned} R(X, Y, Z, W) &= g(R(X, Y)Z, W) = \bar{R}(X, Y, Z, W) \\ &\quad + \bar{g}(\sigma(X, Z), \sigma(Y, W)) - \bar{g}(\sigma(Y, Z), \sigma(X, W)), \end{aligned}$$

$$R^\perp(X, Y, \xi, \eta) = \bar{g}(R^\perp(X, Y)\xi, \eta) = \bar{R}(X, Y, \xi, \eta) - g([A_\xi, A_\eta]X, Y),$$

where $[A_\xi, A_\eta] = A_\xi \circ A_\eta - A_\eta \circ A_\xi$, for all $X, Y, Z, W \in TM$ and $\xi, \eta \in TM^\perp$.

Let $\{e_1, \dots, e_n, e_1^* = Ie_1, \dots, e_n^* = Ie_n\}$ be a local orthonormal frame on CP^n such that, restricted to M , the vector fields $e_1, \dots, e_n$ are tangent to M . We put $A_{i^*} = A_{e_i^*}$, $R_{ijkh} = R(e_i, e_j, e_k, e_h)$ and $R_{ijk^*h^*}^\perp = R^\perp(e_i, e_j, e_k^*, e_h^*)$. Since

$$\begin{aligned} \bar{R}(X, Y, Z, W) = & \frac{c}{4} \{ \bar{g}(X, Z) \bar{g}(Y, W) - \bar{g}(Y, Z) \bar{g}(X, W) \\ & + 2\bar{g}(X, IY) \bar{g}(Z, IW) + \bar{g}(X, IZ) \bar{g}(Y, IW) \\ & - \bar{g}(Y, IZ) \bar{g}(X, IW) \}, \end{aligned}$$

the Gauss and Ricci equations become

$$\begin{aligned} R_{ijkh} = & \frac{c}{4} (\delta_{ik} \delta_{jh} - \delta_{jk} \delta_{ih}) \\ & + \bar{g}(\sigma(e_i, e_k), \sigma(e_j, e_h)) - \bar{g}(\sigma(e_j, e_k), \sigma(e_i, e_h)), \end{aligned} \quad (2.1)$$

and

$$R_{ijk^*h^*}^\perp = \frac{c}{4} (\delta_{ik} \delta_{jh} - \delta_{jk} \delta_{ih}) - g([A_{k^*}, A_{h^*}]e_i, e_j). \quad (2.2)$$

The mean curvature vector is defined by

$$H = \text{trace}(\sigma) = \sum_i \sigma(e_i, e_i) = \sum_i (\text{tr } A_{i^*}) e_i^*.$$

M is said to be *minimal* if $H = 0$, *totally geodesic* if $\sigma = 0$, *parallel* (or *with parallel second fundamental form*) if $\nabla' \sigma = 0$, where

$$(\nabla'_X \sigma)(Y, Z) = \nabla_X^\perp(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).$$

The scalar curvature of M is given by

$$\tau = n(n-1) \frac{c}{4} + \|H\|^2 - \|\sigma\|^2, \quad (2.3)$$

where $\|\sigma\|^2 = \sum \text{tr } A_{i^*}^2$ and $\|H\|^2 = \sum (\text{tr } A_{i^*})^2$.

We shall make use of the following

Lemma 2.1 ([13]). *Let M be an n -dimensional totally real submanifold of CP^n . Then*

$$\|R\|^2 = c\tau - 2n(n-1) \frac{c^2}{16} - \sum_{i,j} \text{tr}[A_{i^*}, A_{j^*}]^2. \quad (2.4)$$

If in addition M is minimal, then

$$\|\varrho\|^2 = 2(n-1)\frac{c}{4}\tau - n(n-1)^2\frac{c^2}{16} + \sum (\text{tr } A_{i*}A_{j*})^2, \quad (2.5)$$

$$\frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 - \|R\|^2 - \|\varrho\|^2 + (n+1)\frac{c}{4}\tau \quad (2.6)$$

where ϱ is the Ricci tensor of M .

3. Totally real parallel Einstein submanifolds of CP^n

H. NAITOH [11] classified n -dimensional totally real parallel submanifolds of CP^n . We now recall some basic ideas of [11], in order to determine all n -dimensional totally real parallel Einstein submanifolds of CP^n .

Fix an n -dimensional simply connected Riemannian symmetric space M . By $\tilde{T}_M$ (respectively, $\tilde{\mathcal{S}}_M$) we denote the set of equivalence classes of totally real parallel isometric immersions of M into CP^n (respectively, of complete totally real parallel submanifolds in CP^n , having M as universal covering).

Since M is symmetric, there exists a Lie group G acting isometrically and transitively on M . M is isometric to a quotient M/K and the Lie algebra $\mathfrak{g}$ of G splits as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, with $\mathfrak{p}$ isometric to the tangent space T_oM at a point o of M . By $\mathcal{M}_M$ it is denoted the set of all $\mathfrak{p}$ -valued bilinear forms $\tilde{\sigma}$ on $\mathfrak{p}$, satisfying

- (1) $\tilde{\sigma}$ is a symmetric trilinear form on $\mathfrak{p}$, under the canonical identification of $\mathfrak{p}^* \otimes \mathfrak{p}^* \otimes \mathfrak{p}$ with $\mathfrak{p}^* \otimes \mathfrak{p}^* \otimes \mathfrak{p}^*$ through the Riemannian metric $\langle \cdot, \cdot \rangle$ on $\mathfrak{p}$,
- (2) $\mathfrak{t} \cdot \tilde{\sigma} = 0$, and
- (3) $\frac{c}{4}(\langle Y, Z \rangle X - \langle X, Z \rangle Y) = R(X, Y)Z - [\tilde{\sigma}(X), \tilde{\sigma}(Y)](Z)$, for all vectors $X, Y, Z \in \mathfrak{p}$.

When $f : M \rightarrow CP^n$ is a totally real parallel isometric immersion, then $(\tilde{\sigma}_f)_o$ belongs to $\mathcal{M}_M$, where σ_f is the second fundamental form associated to f and $\tilde{\sigma}_f$ is defined by

$$\tilde{\sigma}_f(X, Y) = J\sigma_f(X, Y),$$

J being the complex structure of CP^n . A suitable equivalence is introduced in $\mathcal{M}_M$, so that the quotient set $\bar{\mathcal{M}}_M$ of $\mathcal{M}_M$ has a natural one-to-one correspondance with $\bar{\mathcal{T}}_M$ (and so, also with $\bar{\mathcal{S}}_M$). Therefore, the problem of classifying totally real parallel submanifolds of CP^n reduces to the problem of studying $\bar{\mathcal{M}}_M$, for any given simply connected symmetric space M . We refer to [11] for more details.

The following results were proved in [11, Section 4], where M is supposed to be without Euclidean factor.

Lemma 3.1 ([11]). *Assume that $\mathcal{M}_M$ is not empty. Then the simply connected symmetric space M without Euclidean factor is irreducible and of compact type.*

Note that, as it is well-known, an irreducible symmetric Riemannian manifold is Einsteinian.

Theorem 3.2 ([11]). *Let M be a simply connected symmetric space without Euclidean factor. Then the set $\bar{\mathcal{M}}_M$ is not empty if and only if M is one of the followings:*

$$\begin{aligned} &SO(n+1)/SO(n) \ (n \geq 2), \quad SU(k), \ (k \geq 3), \\ &SU(k)/SO(k), \ (k \geq 3), \quad SU(2k)/Sp(k), \ (k \geq 3), \quad E_6/F_4. \end{aligned} \tag{3.1}$$

In this case, the metric on M is determined uniquely by the constant c (the holomorphic sectional curvature of CP^n) and the set $\bar{\mathcal{M}}_M$ consists of one point.

Note that $SO(n+1)/SO(n)$ is the Euclidean sphere $S^n(\beta)$, for some $\beta > 0$.

Suppose now that M is an n -dimensional totally real parallel Einstein submanifold of $CP^n(c)$.

a) If M has no Euclidean factor, then M is one of the spaces listed in Theorem 3.2. Note that all these spaces are compact and their immersions in $CP^n(c)$ are minimal [11, Remark 5.3].

We can determine explicitly the metric of the Einstein submanifolds of $CP^n(c)$ listed in (3.1). As claimed in Theorem 3.2, their metrics are determined by c . In fact, since M is parallel and compact, integrating

(2.6) we get

$$\int_M \|R\|^2 dv + \int_M \|\varrho\|^2 dv - (\dim M + 1) \frac{c}{4} \int_M \tau dv = 0,$$

from which, if $\|R\|^2$ is constant (τ and $\|\varrho\|^2 = \tau^2 / \dim M$ are constant, M being an Einstein manifold), it follows

$$\|R\|^2 + \frac{1}{\dim M} \tau^2 - (\dim M + 1) \frac{c}{4} \tau = 0. \quad (3.2)$$

Curvature invariants of symmetric spaces of rank 1 and of classical symmetric spaces were calculated in [7] (some corrections were successively needed and they have been made in [5]). In particular, for the spaces listed in (3.1), we have

M	$\dim$	τ	$\ R\ ^2$
$S^n(\beta)$	n	$n(n-1)\beta$	$2n(n-1)\beta^2$
$SU(k)$	$k^2 - 1$	$4k(k^2 - 1)\beta$	$16k^2(k^2 - 1)\beta^2$
$SU(k)/SO(k)$	$\frac{1}{2}(k-1)(k+2)$	$k(k-1)(k+2)\beta$	$2k(k-1)(k+2)^2\beta^2$
$SU(2k)/Sp(k)$	$(k-1)(2k+1)$	$4k(k-1)(2k+1)\beta$	$16k(k-1)^2(2k+1)\beta^2$

Table I

The metric on M is defined up to a homothetic transformation and so, curvature invariants of M depend on $\beta > 0$. We did not report the value of $\|\varrho\|^2$ since M is an Einstein space and so, $\|\varrho\|^2 = \tau^2 / \dim M$. Using (3.2), we can determine β for such spaces, in function of c . We get

M	$S^n(\beta)$	$SU(k)$	$SU(k)/SO(k)$	$SU(2k)/Sp(k)$
β	$\frac{c}{4}$	$\frac{c}{16k}$	$\frac{kc}{32}$	$\frac{kc}{16}$

Table II

The Riemannian curvature invariants τ and $\|R\|^2$ of M can now be calculated from the above Table I, using the values of β listed in Table II.

For what concerns E_6/F_4 , it was noted in [11, Remark 5.4] that its immersion f in CP^n is $\frac{\sqrt{c}}{2\sqrt{2}}$ -isotropic, that is, $\sigma_f(X.X) = \frac{\sqrt{c}}{2\sqrt{2}}$ for any unit tangent vector X of M . In particular, this implies that $\|\sigma_f\|^2 = \dim M \frac{1}{8}c = \frac{13}{4}c$ and so, by (2.3), we get $\tau = \frac{637}{4}c$. For E_6/F_4 , being an Einstein space, we have $\|\varrho\|^2 = \tau^2 / \dim M = \frac{31213}{32}c^2$. Finally, using (3.2) we can also compute $\|R\|^2$ and we get $\|R\|^2 = \frac{3185}{32}c^2$. In this way, we determined all n -dimensional totally real parallel Einstein submanifolds of $CP^n(c)$ without Euclidean factor, and calculated explicitly their Riemannian curvature invariants τ , $\|\varrho\|^2$ and $\|R\|^2$.

b) Suppose now that M is an n -dimensional totally real parallel Einstein submanifold of $CP^n(c)$, having a Euclidean factor. Therefore, we have

$$M = \mathbb{R}^{n_0} \times M_1^{n_1} \times \cdots \times M_r^{n_r},$$

with $n = \sum_{j=0}^r n_j$, $n_0 > 0$, and $M_i^{n_i}$ is an n_i -dimensional irreducible simply connected symmetric space for each i [11].

In our case, since M is an Einstein space given by a Riemannian product of Einstein spaces, we must have

$$0 = \frac{\tau_0}{n_0} = \frac{\tau_1}{n_1} = \cdots = \frac{\tau_r}{n_r},$$

that is, $\tau_i = 0$ for all i . But none of the spaces listed in a) has zero scalar curvature. Therefore, if M has a Euclidean factor, then M itself is Euclidean. In particular, if M is compact, then M is the n -dimensional flat torus, T^n .

Therefore, we proved the following

Theorem 3.3. *Let M be an n -dimensional totally real parallel Einstein submanifold of the complex projective space $CP^n(c)$. If M has no Euclidean factor, then M is one of the spaces listed in (3.1), equipped with a Riemannian metric uniquely determined by c . In particular, M is compact and its immersion in $CP^n(c)$ is minimal. If M has an Euclidean factor, then M is flat (in particular, if M is compact, then $M = T^n$).*

The following Table III describes all n -dimensional compact totally real parallel Einstein submanifolds of $CP^n(c)$.

M	$\dim$	τ	$\ R\ ^2$
$S^n(\frac{c}{4})$	n	$\frac{n(n-1)}{4}c$	$\frac{n(n-1)}{8}c^2$
$SU(k)$	$k^2 - 1$	$\frac{(k^2-1)}{4}c$	$\frac{(k^2-1)^2}{16}c^2$
$SU(k)/SO(k)$	$\frac{1}{2}(k-1)(k+2)$	$\frac{k^2(k-1)(k+2)}{32}c$	$\frac{k^3(k-1)(k+2)^2}{512}c^2$
$SU(2k)/Sp(k)$	$(k-1)(2k+1)$	$\frac{k^2(k-1)(2k+1)}{4}c$	$\frac{k^3(k-1)^2(2k+1)}{16}c^2$
E_6/F_4	26	$\frac{637}{4}c$	$\frac{3185}{32}c^2$
T^n	n	0	0

Table III

It is easy to check that for two of such manifolds, having the same dimension, it never occurs that the pair of Riemannian curvature invariants $(\tau, \|R\|^2)$ attains the same value. Therefore, we proved the following

Theorem 3.4. *Each compact n -dimensional totally real parallel Einstein submanifold of $CP^n(c)$ is uniquely determined by the pair of Riemannian curvature invariants $(\tau, \|R\|^2)$.*

4. Spectral geometry of J and totally real Einstein submanifolds of $CP^n(c)$

Let M be an n -dimensional Riemannian manifold immersed in a Riemannian manifold $\bar{M}$ of dimension $\bar{n} = n + r$. The normal bundle $TM^\perp$ is a real r -dimensional vector bundle on M , with inner product induced by the metric $\bar{g}$ of $\bar{M}$. Let D denote the so-called *rough Laplacian* associated to the normal connection $\nabla^\perp$ of $TM^\perp$, that is,

$$D\xi = -\nabla_{e_i}^\perp \nabla_{e_i}^\perp \xi + \nabla_{\nabla_{e_i} e_i}^\perp \xi,$$

where ξ is a section of $TM^\perp$. Next, let $\tilde{A}$ be the *Simons operator* defined in [15] by

$$\bar{g}(\tilde{A}\xi, \eta) = \text{tr}(A_\xi \circ A_\eta),$$

for $\xi, \eta \in TM^\perp$. Moreover, we consider the operator $\tilde{R}$ defined by

$$\tilde{R}(\xi) = - \sum_{i=1}^n (\bar{R}(e_i, \xi)e_i)^\perp,$$

where $(\bar{R}(e_i, \xi)e_i)^\perp$ denotes the normal component of $\bar{R}(e_i, \xi)e_i$.

The *Jacobi operator* (or *second variation operator*), acting on cross-sections of $TM^\perp$, is the second order elliptic differential operator J defined by (see [15] or [4])

$$J : TM^\perp \longrightarrow TM^\perp$$

$$\xi \longmapsto (D - \tilde{A} + \tilde{R})\xi.$$

Let $f : M \rightarrow \bar{M}$ be an isometric minimal immersion. A *variation* of f is a one parameter family $\{f_t\}$ of immersions $M \rightarrow \bar{M}$, such that $f_0 = f$ and $F : M \times [0, 1] \rightarrow \bar{M}$, with $F(m, t) = f_t(m)$, is C^∞ . If $\mathcal{A}(t)$ denotes the area associated to f_t , then the Jacobi operator expresses the second variation for $\mathcal{A}$, since

$$\mathcal{A}''(0) = \int_M \langle JV, V \rangle dv$$

(see [15]). Similarly, if $\phi : (M, g) \rightarrow (N, h)$ is a harmonic map and $\{\phi_t\}$ a variation of ϕ , then the Jacobi operator J_ϕ expresses the second variation of the energy $\mathcal{E}(t) = \mathcal{E}(\phi_t)$ associated to ϕ , by

$$\mathcal{E}''(0) = \int_M h(V, J_\phi V) dv$$

(see for example [16]).

When M is compact, we can define an inner product for cross-sections on $TM^\perp$, by

$$\langle \xi, \eta \rangle = \int_M \bar{g}(\xi, \eta) dv$$

and J is self-adjoint with respect to this product. Moreover, J is strongly elliptic and it has an infinite sequence of eigenvalues, with finite multiplicities, denoted by

$$\text{spec}(M, J) = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots + \uparrow \infty\}.$$

The partition function $Z(t) = \sum_{i=1}^{\infty} \exp(-\lambda_i t)$ has the asymptotic expansion

$$Z(t) \sim (4\pi t)^{-n/2} \{a_0(J) + a_1(J)t + a_2(J)t^2 + \dots\}.$$

By GILKEY's results [6] (see also [4] and [8]), the coefficients a_0 , a_1 and a_2 are given by the following

Theorem 4.1 ([6]).

$$\begin{aligned} a_0 &= r \operatorname{vol}(M), \\ a_1 &= \frac{r}{6} \int_M \tau dv + \int_M \operatorname{tr} \tilde{E} dv, \\ a_2 &= \frac{r}{360} \int_M \{2\|R\|^2 - 2\|\varrho\|^2 + 5\tau^2\} dv \\ &\quad + \frac{1}{360} \int_M \{-30\|R^\perp\|^2 + \operatorname{tr}(60\tau\tilde{E} + 180\tilde{E}^2)\} dv, \end{aligned}$$

where $\tilde{E} = \tilde{A} - \tilde{R}$.

In the case of an n -dimensional totally real submanifold of CP^n , the coefficients a_0 , a_1 and a_2 were computed explicitly in [1], in terms of curvature invariants of M . In particular, the following result has been obtained.

Theorem 4.2 ([1]). *On an n -dimensional totally real minimal submanifold M of $CP^n(c)$, the first coefficients of the asymptotic expansion of the partition function of the Jacobi operator are given by*

$$a_0 = n \operatorname{vol}(M), \tag{4.1}$$

$$\begin{aligned} a_1 &= \frac{n-6}{6} \int_M \tau dv + 2n(n+1) \frac{c}{4} \operatorname{vol}(M) \\ &= \frac{6-n}{6} \int_M \|\sigma\|^2 dv + \frac{n}{6} (n^2 + 5n + 18) \frac{c}{4} \operatorname{vol}(M), \end{aligned} \tag{4.2}$$

$$\begin{aligned} a_2 &= \frac{1}{360} \int_M \{2(n-15)\|R\|^2 - 2(n-90)\|\varrho\|^2 \\ &\quad + 5(n-12)\tau^2\} dv + \frac{(n+1)(n-6)}{3} \frac{c}{4} \int_M \tau dv \end{aligned}$$

$$+ 2n(n+1)^2 \frac{c^2}{16} \text{vol}(M). \quad (4.3)$$

In the sequel, we shall denote by M_0 one of the compact totally real submanifolds of $CP^n(c)$ listed in Table III. Our purpose is to characterize M_0 by its $\text{spec}(J)$ in the class of all compact totally real minimal submanifolds of $CP^n(c)$. We first remark that, as an easy consequence of Theorem 3.4, we get the following

Theorem 4.3. *Each compact n -dimensional totally real parallel Einstein submanifold M_0 of $CP^n(c)$ is uniquely determined by its $\text{spec}(J)$.*

PROOF. We treat separately the cases $n \neq 6, 15$, $n = 6$ and $n = 15$.

a) If $n \neq 6, 15$, by Theorem 3.4, it is enough to prove that $\text{spec}(J)$ determines the pair of Riemannian invariants $(\tau, \|R\|^2)$ of M . In fact, suppose that $\text{spec}(M_0, J) = \text{spec}(M'_0, J)$, where M_0, M'_0 are n -dimensional compact totally real Einstein submanifolds of $CP^n(c)$. Then, since $n \neq 6$, (4.1) and (4.2) imply that $\tau_0 = \tau'_0$. M_0, M'_0 being Einstein manifolds having the same dimension, it follows that $\|\varrho_0\|^2 = \|\varrho'_0\|^2$. Thus, since $n \neq 15$, taking into account that $\|R_0\|^2$ and $\|R'_0\|^2$ are constant, from (4.3) we get $\|R_0\|^2 = \|R'_0\|^2$.

b) If $n = 6$, from Table III we see that $M_0 = S^6(\frac{c}{4})$ or $M_0 = T^6$. Suppose that $\text{spec}(S^6(\frac{c}{4}), J) = \text{spec}(T^6, J)$. Then, in particular, $a_0(S^6(\frac{c}{4})) = a_0(T^6)$ and $a_2(S^6(\frac{c}{4})) = a_2(T^6)$, from which it follows easily that c vanishes, which can not occur.

c) If $n = 15$, from Table III we see that $M_0 = S^{15}(\frac{c}{4})$, T^{15} or $SU(4)$. Suppose that $\text{spec}(M_0, J) = \text{spec}(M'_0, J)$. Then, in particular, $a_0(M_0) = a_0(M'_0)$ and $a_1(M_0) = a_1(M'_0)$, from which it follows easily $\tau_0 = \tau'_0$, which can not occur, because, as it follows from Table III, for $S^{15}(\frac{c}{4})$, T^{15} and $SU(4)$ we respectively have $\tau = \frac{105}{2}c$, 0 and $\frac{15}{4}c$, with $c \neq 0$ $\square$

We now prove the following

Theorem 4.4. *Let M be an n -dimensional compact totally real minimal submanifold of $CP^n(c)$. If $\text{spec}(M, J) = \text{spec}(M_0, J)$, $16 \leq \dim M_0 \leq 52$, then M is isometric to M_0 .*

PROOF. Since $\text{spec}(M, J) = \text{spec}(M_0, J)$, we have $\dim M_0 = \dim M = n$ and, from Theorem 4.2,

$$\text{vol}(M, g) = \text{vol}(M_0, g_0), \quad (4.4)$$

$$\int_M \tau dv = \int_{M_0} \tau_0 dv, \quad \int_M \|\sigma\|^2 dv = \int_{M_0} \|\sigma_0\|^2 dv, \quad (4.5)$$

$$\begin{aligned} & \int_M \{2(n-15)\|R\|^2 + 2(90-n)\|\varrho\|^2 + 5(n-12)\tau^2\} dv \\ &= \int_{M_0} \{2(n-15)\|R_0\|^2 + 2(90-n)\|\varrho_0\|^2 + 5(n-12)\tau_0^2\} dv \end{aligned} \quad (4.6)$$

Since τ_0 is constant and $\text{vol}(M) = \text{vol}(M_0)$, we have

$$\begin{aligned} \int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv &= \int_M \tau^2 dv - 2\tau_0 \int_{M_0} \tau_0 dv + \int_{M_0} \tau_0^2 dv \\ &= \int_M (\tau - \tau_0)^2 dv \geq 0 \end{aligned} \quad (4.7)$$

where the equality holds if and only if $\tau = \tau_0$.

Next, let $E = \varrho - \frac{\tau}{n}g$ denote the *Einstein curvature tensor* of (M, g) . Since $\|E\|^2 = \|\varrho\|^2 - \frac{\tau^2}{n}$ and $E_0 = 0$ because M_0 is an Einstein space, (4.6) becomes

$$\begin{aligned} & 2(n-15) \left(\int_M \|R\|^2 dv - \int_{M_0} \|R_0\|^2 dv \right) - 2(n-90) \int_M \|E\|^2 dv \\ &+ \frac{5n^2 - 62n + 180}{n} \left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv \right) = 0. \end{aligned} \quad (4.8)$$

Moreover, from (2.6) we also get

$$\frac{1}{2} \Delta \|\sigma\|^2 = \|\nabla' \sigma\|^2 - \|R\|^2 - \|E\|^2 + \frac{1}{n} \tau^2 + (n+1) \frac{c}{4} \tau.$$

Integrating over M , we obtain

$$\begin{aligned} \int_M \|\nabla' \sigma\|^2 dv &= \int_M \|R\|^2 dv + \int_M \|E\|^2 dv \\ &+ \frac{1}{n} \int_M \tau^2 dv - (n+1) \frac{c}{4} \int_M \tau dv. \end{aligned} \quad (4.9)$$

An analogous formula holds for M_0 , with $\nabla'\sigma_0 = E_0 = 0$. We use (4.9) to calculate $\int_M \|R\|^2 dv$. Therefore, (4.8) becomes

$$(n-15) \int_M \|\nabla'\sigma\|^2 dv = \alpha(n) \int_M \|E\|^2 dv + \beta(n) \left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv \right), \quad (4.10)$$

where

$$\alpha(n) = 2n - 105, \\ \beta(n) = -\frac{5n^2 - 64n + 210}{2n}.$$

It is easy to check that if $16 \leq n \leq 52$, then $n-15 > 0$ while $\alpha(n), \beta(n) < 0$. Therefore, we get $\nabla'\sigma = 0$, $E = 0$ and $\tau = \tau_0$. Thus, M is an Einstein totally real parallel submanifold of $CP^n(c)$ with the same $\text{spec}(J)$ of M_0 . So, Theorem 4.3 implies that M is isometric to M_0 . $\square$

Remark 4.1. Note that formula (4.10) holds for all n -dimensional compact totally real minimal submanifolds M of CP^n such that $\text{spec}(M, J) = \text{spec}(M_0, J)$.

In particular, if M is also Einsteinian, then (4.10) becomes

$$(n-15) \int_M \|\nabla'\sigma\|^2 dv = \beta(n) \left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv \right). \quad (4.11)$$

Since $\beta(n) < 0$ for all $n \geq 3$, proceeding as in the proof of Theorem 4.4, we obtain the following

Theorem 4.5. *In the class of all n -dimensional compact totally real Einstein minimal submanifolds of $CP^n(c)$, the parallel ones are characterized by their $\text{spec}(J)$ for all $n \geq 16$.*

Remark 4.2. If $M_0 = S^n(\frac{c}{4})$, then $\sigma_0 = 0$ and (4.5) gives at once $\sigma = 0$. Therefore:

In the class of compact totally real minimal submanifolds of $CP^n(c)$, $S^n(\frac{c}{4})$ is characterized by its $\text{spec}(J)$ for all $n \neq 6$.

Remark 4.3. In [1], it was proved that in the class of all n -dimensional compact totally real minimal submanifolds of $CP^n(c)$, the parallel conformally flat ones are characterized by their $\text{spec}(J)$ when $53 \leq n \leq 93$. Since the flat torus T^n is at the same time Einstein and conformally flat, combining this result with Theorem 4.4, we obtain the following

Theorem 4.6. *In the class of all n -dimensional compact totally real minimal submanifolds of $CP^n(c)$, the flat torus T^n is characterized by its $\text{spec}(J)$ when $16 \leq n \leq 93$.*

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