

Non-planar simplices are not reduced

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Abstract. A convex body in $\mathbb{R}^n$ which does not properly contain a convex body of the same minimum width is called a reduced body. It is not known whether there exist reduced n -dimensional polytopes for $n \geq 3$. We prove that no n -dimensional simplex is reduced if $n \geq 3$.

1. Introduction

Due to E. HEIL [4], a convex body $K \subset \mathbb{R}^n$ is called *reduced* if there is no convex body L properly contained in K such that the *minimum width* $\Delta(L)$ (=minimal distance between two different parallel supporting hyperplanes) of L is equal to $\Delta(K)$. Reduced bodies are interesting in view of several extremal problems, for example regarding the long-standing question: Which convex body of given minimum width has minimal volume? The extremal body has obviously to be reduced. Every body of constant width in $\mathbb{R}^n$ is reduced, but there are many further examples. For instance,

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all regular m -gons in $\mathbb{R}^2$ with m odd are reduced, as well as the intersection of the unit ball of $\mathbb{R}^n$ with an orthant of the respective Cartesian coordinate system (for $n = 2$ yielding a quarter of the unit disk). Many geometric properties of reduced bodies were found by M. LASSAK [5]. In his paper also the following problem was posed: *Do there exist reduced n -dimensional polytopes for $n \geq 3$?*

Although this question was repeated in [6], the answer is still unknown. Using special geometric properties of tetrahedra (that no longer hold for n -simplices if $n \geq 4$), the authors of [9] proved that there is no reduced 3-simplex. It is our goal to extend this observation to higher dimensions.

2. The result and its proof

For an n -dimensional simplex $S \subset \mathbb{R}^n$, $n \geq 2$, we will use the following notions and abbreviations. The *vertex set* of S is given by $\{x_1, \dots, x_{n+1}\}$ and, for any $i \in \{1, \dots, n+1\}$, F_i denotes the unique $(n-1)$ -face of S which is opposite to the vertex x_i . We also use some functions defined on the unit sphere S^{n-1} of $\mathbb{R}^n$. Most of their properties considered here hold for arbitrary convex bodies (see [3]), but we introduce them only for simplices. For an arbitrary unit vector $u \in S^{n-1}$ the *width* $w(S, u)$ of S in direction u is the distance of the two different parallel supporting hyperplanes of S which are orthogonal to u . The minimum of the function $w(S, u)$, $u \in S^{n-1}$, is called the *minimum width* or *thickness* of S , and is denoted by $\Delta(S)$. There exists a chord of S parallel to the direction of that minimum and having length $\Delta(S)$ (see [2, §§ 33]). Such a chord is said to be a *thickness chord* of S . Thus, if a segment $[a, b] \subset S$ is a thickness chord of S , then there are different supporting hyperplanes H_1, H_2 of S which are both orthogonal to $[a, b]$ and satisfy $a \in H_1, b \in H_2$. In other words, denoting by $V_1(S, u)$, $u \in S^{n-1}$, the function describing the maximal chord length of S for any direction u , we have

$$\min_{u \in S^{n-1}} V_1(S, u) = \Delta(S). \quad (1)$$

The *brightness function* $V_{n-1}(S, u)$, $u \in S^{n-1}$, of an n -simplex S is the $(n-1)$ -volume of the orthogonal projection of S onto the $(n-1)$ -subspace orthogonal to u .

In [8] it was shown that for the *volume* $V_n(S)$ of an arbitrary n -simplex S and *any* direction $u \in S^{n-1}$ the relation

$$V_n(S) = \frac{1}{n} \cdot V_{n-1}(S, u) \cdot V_1(S, u) \quad (2)$$

holds. With (1) this implies in particular

$$V_n(S) = \frac{1}{n} \cdot \max_{u \in S^{n-1}} V_{n-1}(S, u) \cdot \Delta(S), \quad (3)$$

i.e., the maximum brightness and the minimum width of S occur in the same direction.

Now we are ready to prove our

Theorem. *No n -dimensional simplex $S \subset \mathbb{R}^n$, $n \geq 3$, is reduced.*

PROOF. We will prove that statement by contradiction. Assuming that S is reduced, it follows firstly that S has to be *equiareal*, i.e., that each $(n-1)$ -face F_i must have the same $(n-1)$ -volume $V_{n-1}(F_i)$, $i = 1, \dots, n+1$. Indeed, in the classical formula

$$V_n(S) = \frac{1}{n} \cdot V_{n-1}(F_i) \cdot h_i, \quad i \in \{1, \dots, n+1\}, \quad (4)$$

where h_i denotes the length of the i -th altitude of S orthogonal to the affine hull of F_i , h_i is equal to $w(S, u_i)$ with u_i as (outer) normal direction of F_i . If we had $h_i \neq \Delta(S)$ for some $i \in \{1, \dots, n+1\}$, the corresponding vertex x_i would not belong to a thickness chord of S and could be cut off to get from S a convex body L properly contained in S and satisfying $\Delta(L) = \Delta(S)$, a contradiction to the assumed reducedness of S . Thus we must have $h_i = \Delta(S)$ for all $i \in \{1, \dots, n+1\}$, implying by (4) that S is equiareal.

Moreover, combining (4) and (3), we obtain

$$V_{n-1}(F_i) = \max_{u \in S^{n-1}} V_{n-1}(S, u), \quad i = 1, \dots, n+1. \quad (5)$$

From [3, § 4.1] we read off that the brightness function of S has the representation

$$V_{n-1}(S, u) = \frac{1}{2} \sum_{i=1}^{n+1} |\langle v_i, u \rangle|, \quad u \in S^{n-1}, \quad (6)$$

where $v_i := V_{n-1}(F_i) \cdot u_i$. Due to $\sum_{i=1}^{n+1} v_i = o$ (Minkowski's existence theorem, cf. [3, Appendix A]) this can also be written in the form

$$V_{n-1}(S, u) = \sum_{i \in I(u)} \langle v_i, u \rangle, \quad u \in S^{n-1}, \quad (7)$$

where $I(u) := \{j \in \{1, \dots, n+1\} : \langle v_j, u \rangle \geq 0\}$. From (7) it follows that

$$\max_{u \in S^{n-1}} V_{n-1}(S, u) = \left\| \sum_{i \in I^*} v_i \right\|, \quad (8)$$

where the nonempty index set $I^* \subseteq \{1, \dots, n+1\}$ is determined by

$$\left\| \sum_{i \in I^*} v_i \right\| = \max_{I \subseteq \{1, \dots, n+1\}} \left\| \sum_{i \in I} v_i \right\|.$$

Without loss of generality, we may consider $\{v_1, \dots, v_{n+1}\}$ as a system of *unit vectors* since S is assumed to be equiareal. Therefore we can continue with the following

Lemma. *Given $m > 3$ unit vectors $v_1, \dots, v_m$ in $\mathbb{R}^n$. Then there exist distinct indices i, j such that $\|v_i + v_j\| > 1$.*

PROOF. Suppose that $\|v_i + v_j\| \leq 1$ for all $1 \leq i < j \leq m$. Squaring we obtain $\|v_i\|^2 + 2\langle v_i, v_j \rangle + \|v_j\|^2 \leq 1$, implying $2\langle v_i, v_j \rangle \leq -1$. Hence,

$$\left\| \sum_{i=1}^m v_i \right\|^2 = \sum_{i=1}^m \|v_i\|^2 + 2 \sum_{i < j} \langle v_i, v_j \rangle \leq m - \binom{m}{2},$$

yielding $m - \binom{m}{2} \geq 0$. Thus $m \leq 3$, contradicting the hypothesis. $\square$

In view of (8), this lemma says that for an equiareal n -simplex S , $n \geq 3$, the quantity $\max_{u \in S^{n-1}} V_{n-1}(S, u)$ cannot be equal to the $(n-1)$ -volume of an $(n-1)$ -face, i.e., (5) is not satisfied. By (3) it follows that no such simplex has its minimum width in the normal direction of an $(n-1)$ -face, i.e., its vertices are not contained in thickness chords and can be cut off without decreasing $\Delta(S)$. Thus, there is no reduced n -simplex for $n \geq 3$. $\square$

3. Concluding remarks

- (1) Our theorem might be considered as a starting point to solve M. Lasak's problem for all convex n -polytopes (e.g. by some inductual approach based on the cardinality of the vertex set). However, the method presented here can no longer be used. Namely, the function $V_{n-1}(S, u)$, $u \in S^{n-1}$, considered above is known to be the support function of the so-called *projection body* ΠS of the simplex S , and $V_1(S, u)$, $u \in S^{n-1}$, is the radius function of the *difference body* $DS = S + (-S)$ of S . In these terms, relation (2) says that ΠS and DS are polar reciprocal with respect to the sphere of radius $\sqrt{n \cdot V_n(S)}$ which is centred at the origin. (For definitions and many properties of the bodies ΠS and DS , associated with S , the reader should consult [2, §§ 30 and §§ 33] and [3, § 4.1 and § 3.2].) It was proved in [7] that for all convex n -polytopes which are not simplices such a polarity (even with respect to spheres of arbitrary radii) does no longer hold. Thus our conclusion from (2) to (3) is, in general, no longer true.
- (2) To get a dualization of the famous Jung theorem (cf. [2], §§ 44), W. Blaschke erroneously assumed that the minimum width of a *regular* n -simplex in $\mathbb{R}^n$ is attained at the normal directions of its $(n-1)$ -faces, see [1]. (Blaschke's assumption is true only for $n=2$, and his statement for higher dimensions was corrected by P. STEINHAGEN [10].) From our considerations it follows that *no equiareal n -simplex*, $n \geq 3$, has the property assumed by Blaschke.

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