

Salem numbers and uniform distribution modulo 1

By SHIGEKI AKIYAMA (Niigata) and YOSHIO TANIGAWA (Nagoya)

Abstract. For a Salem number α of degree d , the distribution of fractional parts of α^n ($n = 1, 2, \dots$) is studied. By giving explicit inequalities, it is shown to be ‘exponentially’ close to uniform distribution when d is large.

1. Introduction

Uniform distribution of sequences of exponential order growth is an attractive and mysterious subject. Koksma’s Theorem assures that the sequence (α^n) ($n = 0, 1, \dots$) is uniformly distributed modulo 1 for almost all $\alpha > 1$. See [6]. To find an example of such α has been an open problem for a long time. In [7], M. B. LEVIN constructed an $\alpha > 1$ with more strong distribution properties. His method gives us a way to approximate such α step by step. (See also [4, pp. 118–130].) However, no ‘concrete’ examples of such α are known to date. For instance, it is still an open problem whether (e^n) and $((3/2)^n)$ are dense or not in $\mathbb{R}/\mathbb{Z}$ (c.f. BEUKERS [2]).

On the other hand, one can easily construct $\alpha > 1$ that (α^n) is *not* uniformly distributed modulo 1. A Pisot number gives us such an example. We recall the definition of Pisot and Salem numbers. A *Pisot number* is

Mathematics Subject Classification: 11K06, 11J71.

Key words and phrases: Pisot number, Salem number, Distribution modulo 1, Selberg polynomial.

Authors were supported by the Japanese Ministry of Education, Culture, Sports, Science and Technology, Grant-in Aid for fundamental research 14540015 and 14540021, 2002–2005.

a real algebraic integer greater than 1 whose conjugates other than itself have modulus less than 1. A *Salem number* is a real algebraic integer greater than 1 whose conjugates other than itself have modulus less than or equal to 1 and at least one conjugate has modulus equal to 1. It is shown that (α^n) tends to 0 in $\mathbb{R}/\mathbb{Z}$ when α is a Pisot number. If α is a Salem number, (α^n) is dense in $\mathbb{R}/\mathbb{Z}$ but not uniformly distributed modulo 1. (See [1, pp. 87–89].) Moreover, Salem numbers are the only known ‘concrete’ numbers whose powers are dense in $\mathbb{R}/\mathbb{Z}$.

In this short note, we will consider a quantitative problem:

How far is the sequence (α^n) from the uniform distribution for a Salem number α ?

Let (a_n) , $n = 0, 1, \dots$ be a real sequence and I be an interval in $[0, 1]$. Define a counting function $A_N((a_n), I)$ by the cardinality of $n \in \mathbb{Z} \cap [1, N]$ such that $\{a_n\}$, the fractional part of a_n , lie in I . We shall show

Theorem 1. *Let α be a Salem number of degree greater than or equal to 8. Then $\lim_{N \rightarrow \infty} \frac{1}{N} A_N((\alpha^n), I)$ exists and satisfies*

$$\left| \lim_{N \rightarrow \infty} \frac{1}{N} A_N((\alpha^n), I) - |I| \right| \leq 2\zeta\left(\frac{\deg \alpha - 2}{4}\right) (2\pi)^{1 - \frac{\deg \alpha}{2}} |I|,$$

where $\zeta(s)$ is the Riemann zeta function, $\deg \alpha$ is the degree of α over $\mathbb{Q}$ and $|I|$ is the length of I .

Theorem 2. *Let α be a Salem number of degree 4 or 6. Then $\lim_{N \rightarrow \infty} \frac{1}{N} A_N((\alpha^n), I)$ exists and satisfies*

$$\left| \lim_{N \rightarrow \infty} \frac{1}{N} A_N((\alpha^n), I) - |I| \right| \leq 4\pi^{-\frac{3}{2}} \sqrt{|I|} \quad \text{for } \deg \alpha = 4,$$

and

$$\left| \lim_{N \rightarrow \infty} \frac{1}{N} A_N((\alpha^n), I) - |I| \right| \leq \frac{|I|}{2\pi^2} \left(\log \frac{1}{|I|} + 1 + |I| \right) \quad \text{for } \deg \alpha = 6.$$

These theorems show that the sequence (α^n) is quite ‘near’ to uniformly distributed sequences when the degree of a Salem number α is large.

2. Proof of Theorem 1

Let α be a Salem number of degree s . From the definition of Salem numbers, s is an even integer not less than 4, whose conjugates are

$$\alpha, \alpha^{-1}, \alpha^{(1)}, \dots, \alpha^{(s-2)}$$

with complex $\alpha^{(j)}$ of modulus 1 [1, p. 85]. Assume that $\alpha^{(j+r)} = \overline{\alpha^{(j)}}$ for $j = 1, \dots, r$ with $r = \frac{s-2}{2}$. Put

$$\alpha^{(j)} = \exp(2\pi i \theta_j) \quad (0 < \theta_j < 1) \quad (1)$$

for $1 \leq j \leq r$.

Lemma 1. *Let θ_j be the numbers defined by (1). Then $1, \theta_1, \dots, \theta_r$ are linearly independent over $\mathbb{Q}$.*

PROOF. See for example [1, pp. 88–89]. $\square$

From this lemma, $\{(m\theta_1, m\theta_2, \dots, m\theta_r)\}_{m=1}^\infty$ is uniformly distributed mod $\mathbb{Z}^r$. Hence for any Riemannian integrable function $f(x)$ on $(\mathbb{R}/\mathbb{Z})^r$, the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{m=1}^N f(m\theta_1, \dots, m\theta_r)$$

exists and is equal to

$$\int_{(\mathbb{R}/\mathbb{Z})^r} f(x_1, \dots, x_r) dx_1 \cdots x_r.$$

Let $I = [a, b]$ be an interval in $[0, 1]$ and χ_I the characteristic function of I . We extend χ_I as a periodic function on $\mathbb{R}$ by a period 1. Since $A_N((\alpha^n), I) = \sum_{m=1}^N \chi_I(\alpha^m)$ and

$$\alpha^m + \alpha^{-m} + 2 \sum_{j=1}^r \cos(2\pi m \theta_j) \in \mathbb{Z},$$

we study the limit of

$$S_N(\alpha, I) := \frac{1}{N} \sum_{m=1}^N \chi_I \left(-\alpha^{-m} - 2 \sum_{j=1}^r \cos(2\pi m \theta_j) \right) \quad (2)$$

as $N \rightarrow \infty$.

For that purpose, we recall the Selberg polynomial which approximates the characteristic function of an interval. Let $\Delta_K(x)$ be the Fejér's kernel defined by

$$\Delta_K(x) = 1 + \sum_{\substack{|k| \leq K \\ k \neq 0}} \left(1 - \frac{|k|}{K}\right) e^{2\pi i k x},$$

and $V_K(x)$ be the Vaaler's polynomial:

$$V_K(x) = \frac{1}{K+1} \sum_{k=1}^K f\left(\frac{k}{K+1}\right) \sin(2\pi k x)$$

where $f(u) = -(1-u) \cot(\pi u) - \frac{1}{\pi}$. It is clear that for any η ($0 < \eta \leq 1/2$),

$$|f(u)| \leq \begin{cases} \frac{\pi\eta}{\sin \pi\eta} \frac{1}{\pi u} + \frac{1}{\pi} & \text{for } 0 < u \leq \eta \\ \frac{1-\eta}{\sin \pi(1-\eta)} + \frac{1}{\pi} & \text{for } \eta < u < 1. \end{cases} \quad (3)$$

Furthermore let $B_K(x)$ denote the Beurling polynomial:

$$B_K(x) = V_K(x) + \frac{1}{2(K+1)} \Delta_{K+1}(x). \quad (4)$$

Take an interval $J = [a, b]$ in $[0, 1]$. Then Selberg polynomials for the interval J are

$$S_K^+(x) = b - a + B_K(x - b) + B_K(a - x) \quad (5)$$

and

$$S_K^-(x) = b - a - B_K(b - x) - B_K(x - a). \quad (6)$$

These functions $S_K^\pm(x)$ are trigonometric polynomials of degree at most K and satisfy

$$S_K^-(x) \leq \chi_J(x) \leq S_K^+(x). \quad (7)$$

See [8] for further properties of Selberg polynomials.

Lemma 2. *Let k be a positive integer. Then we have*

$$|J_0(2\pi k)| \leq \frac{1}{\pi\sqrt{2k}}. \quad (8)$$

PROOF. Let $H_\nu^{(j)}(z)$ ($j = 1, 2$) be the Hankel functions. Asymptotic expansions of $H_\nu^{(j)}(z)$ are given by

$$H_\nu^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i(z - \frac{\nu\pi}{2} - \frac{\pi}{4})} \left\{ \sum_{m=0}^{p-1} \frac{(-1)^m(\nu, m)}{(2iz)^m} + R_p^{(1)}(z) \right\}$$

and

$$H_\nu^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i(z - \frac{\nu\pi}{2} - \frac{\pi}{4})} \left\{ \sum_{m=0}^{p-1} \frac{(\nu, m)}{(2iz)^m} + R_p^{(2)}(z) \right\},$$

where $(\nu, m) = \frac{(4\nu^2-1)(4\nu^2-3^2)\dots(4\nu^2-(2m-1)^2)}{2^{2m}m!}$, $(\nu, 0) = 1$ and $R_p^{(j)}(z)$ ($j = 1, 2$) are remainder terms ([9, pp. 197–198]). Taking $\nu = 0$, $p = 2$, we get

$$\begin{aligned} J_\nu(z) &= \frac{1}{2} \left(H_\nu^{(1)}(2\pi k) + H_\nu^{(2)}(2\pi k) \right) \\ &= \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left\{ \cos\left(z - \frac{\pi}{4}\right) + \frac{1}{8z} \sin\left(z - \frac{\pi}{4}\right) + \frac{1}{2} \left(R_2^{(1)}(z) + R_2^{(2)}(z) \right) \right\}. \end{aligned}$$

It is easily seen that for $j = 1, 2$

$$|R_2^{(j)}(z)| \leq \frac{9}{128z^2} \quad \text{for } z > 0$$

(see the integral representation of $R_p^{(j)}(z)$ in [9, p. 197]). Hence

$$J_0(2\pi k) = \frac{1}{\pi\sqrt{k}} \left(\frac{1}{\sqrt{2}} - \frac{1}{16\sqrt{2}\pi k} + R \right)$$

with

$$\begin{aligned} |R| &\leq \frac{1}{2} \left(|R_2^{(1)}(2\pi k)| + |R_2^{(2)}(2\pi k)| \right) \leq \frac{9}{512\pi^2 k^2} \\ &\leq \frac{1}{16\sqrt{2}\pi k}, \end{aligned}$$

we get the assertion of the lemma. $\square$

Lemma 3. Take a and b in $[0, 1]$ with $a < b$ and let $J = (a, b), [a, b], (a, b]$ or $[a, b)$. Let r be an integer not less than 3. Then we have

$$\left| \int_{(\mathbb{R}/\mathbb{Z})^r} \chi_J \left(-2 \sum_{j=1}^r \cos(2\pi x_j) \right) dx_1 \cdots dx_r - |J| \right| \leq 2\zeta\left(\frac{r}{2}\right)(2\pi)^{-r}|J|. \quad (9)$$

PROOF. Hereafter we write $z = 2 \sum_{j=1}^r \cos(2\pi x_j)$ and $W = (\mathbb{R}/\mathbb{Z})^r$ for simplicity. By (7), we evaluate the integrals:

$$\int_W \left\{ B_K(\mp(z+b)) + B_K(\pm(z+a)) \right\} dx_1 \cdots dx_r. \quad (10)$$

Substituting (4), the definition of $B_K(x)$, and using the integral formula

$$\begin{aligned} \int_W e^{\pm 2\pi i k(z+a)} dx_1 \cdots dx_r &= e^{\pm 2\pi i k a} \left(\int_0^1 e^{4\pi i k \cos 2\pi x} dx \right)^r \\ &= e^{\pm 2\pi i k a} J_0(4\pi k)^r, \end{aligned}$$

(see [5, p. 81]), we have

$$\begin{aligned} \int_W B_K(z+a) dx_1 \cdots dx_r &= \int_W \left\{ V_K(z+a) + \frac{\Delta_{K+1}(z+a)}{2(K+1)} \right\} dx_1 \cdots dx_r \\ &= \frac{1}{K+1} \sum_{k=1}^K f\left(\frac{k}{K+1}\right) \sin(2\pi k a) J_0(4\pi k)^r \\ &\quad + \frac{1}{2(K+1)} \left\{ 1 + \sum_{\substack{|k| \leq K+1 \\ k \neq 0}} \left(1 - \frac{|k|}{K+1} \right) e^{2\pi i k a} J_0(4\pi k)^r \right\}. \end{aligned} \quad (11)$$

From (8) the absolute value of the last term on the right hand side of (11) is estimated as

$$\begin{aligned} &\leq \frac{1}{2(K+1)} \left\{ 1 + 2(2\pi)^{-r} \sum_{k=1}^{K+1} \left(1 - \frac{k}{K+1} \right) k^{-r/2} \right\} \\ &\leq \frac{1}{2(K+1)} \left\{ 1 + 2(2\pi)^{-r} \zeta\left(\frac{r}{2}\right) \right\} \leq \frac{1}{K}. \end{aligned}$$

Hence the integral of $B_K(z + a)$ is given by

$$\begin{aligned} & \int_W B_K(z + a) dx_1 \cdots dx_r \\ &= \frac{1}{K+1} \sum_{k=1}^K f\left(\frac{k}{K+1}\right) \sin(2\pi ka) J_0(4\pi k)^r + G_1(a) \end{aligned}$$

with the bound $|G_1(a)| \leq \frac{1}{K}$. The integral of $B_K(-z - b)$ is given in the same way,

$$\begin{aligned} & \int_W B_K(-z - b) dx_1 \cdots dx_r \\ &= -\frac{1}{K+1} \sum_{k=1}^K f\left(\frac{k}{K+1}\right) \sin(2\pi kb) J_0(4\pi k)^r + G_2(b) \end{aligned}$$

with the same upper bound $|G_2(b)| \leq \frac{1}{K}$. Adding the above expressions we have

$$\begin{aligned} & \left| \int_W \left\{ B_K(-z - b) + B_K(z + a) \right\} dx_1 \cdots dx_r \right| \\ & \leq \left| \frac{1}{K+1} \sum_{k=1}^K f\left(\frac{k}{K+1}\right) (\sin 2\pi ka - \sin 2\pi kb) J_0(4\pi k)^r \right| + \frac{2}{K} \\ & \leq \frac{2}{K+1} \sum_{k=1}^K \left| f\left(\frac{k}{K+1}\right) \right| |\sin \pi k(a - b)| (2\pi)^{-r} k^{-\frac{r}{2}} + \frac{2}{K} \\ & \leq \frac{(2\pi)^{1-r}}{K+1} (b - a) \sum_{k=1}^K \left| f\left(\frac{k}{K+1}\right) \right| k^{1-\frac{r}{2}} + \frac{2}{K}. \end{aligned}$$

Now we estimate the sum in the above equation. Let ε be a small positive number, and take $\eta < \frac{1}{2}$ to be a small positive number which satisfies $\frac{\pi\eta}{\sin \pi\eta} < 1 + \varepsilon$. Dividing the sum into two parts at $[\eta(K+1)]$ and using (3), we have

$$\frac{1}{K+1} \sum_{k=1}^K \left| f\left(\frac{k}{K+1}\right) \right| k^{1-\frac{r}{2}} \leq \frac{1}{K+1} \sum_{k=1}^{[\eta(K+1)]} \left(\frac{\pi\eta}{\sin \pi\eta} \frac{K+1}{\pi k} + \frac{1}{\pi} \right) k^{1-\frac{r}{2}}$$

$$\begin{aligned}
& + \frac{1}{K+1} \left(\frac{1-\eta}{\sin \pi(1-\eta)} + \frac{1}{\pi} \right) \sum_{k=[\eta(K+1)]+1}^K k^{1-\frac{\tau}{2}} \\
& \leq \frac{1}{\pi} (1+\varepsilon) \zeta\left(\frac{r}{2}\right) + O\left(\frac{1}{\sqrt{K}}\right),
\end{aligned}$$

where the implied constant in the last equation does not depend on K . Therefore

$$\begin{aligned}
& \left| \int_W \left\{ B_K(-z-b) + B_K(z+a) \right\} dx_1 \cdots dx_r \right| \\
& \leq 2(2\pi)^{-r} (b-a) (1+\varepsilon) \zeta\left(\frac{r}{2}\right) + O\left(\frac{1}{\sqrt{K}}\right).
\end{aligned}$$

In the same manner we have

$$\begin{aligned}
& \left| \int_W \left\{ B_K(z+b) + B_K(-z-a) \right\} dx_1 \cdots dx_r \right| \\
& \leq 2(2\pi)^{-r} (b-a) (1+\varepsilon) \zeta\left(\frac{r}{2}\right) + O\left(\frac{1}{\sqrt{K}}\right).
\end{aligned}$$

Thus from (5), (6) and (7) we get the upper bound of the left hand side of (9):

$$\begin{aligned}
& \left| \int_W \chi_J \left(-2 \sum_{j=1}^r \cos(2\pi x_j) \right) dx_1 \cdots dx_r - |J| \right| \\
& \leq 2(1+\varepsilon) \zeta\left(\frac{r}{2}\right) (2\pi)^{-r} |J| + O\left(\frac{1}{\sqrt{K}}\right).
\end{aligned}$$

Now we let $K \rightarrow \infty$, as ε is arbitrary, we get the assertion of the lemma. $\square$

PROOF OF THEOREM 1. Now we study $\lim_{N \rightarrow \infty} S_N(\alpha, I)$ of (2). Let (x_n) and (y_n) be real sequences with $y_n \rightarrow 0$. Then it is easily seen from [6], Chapter 1, Theorem 7.3 that if (x_n) has a continuous asymptotic density function, then $(x_n + y_n)$ also does and their density functions are the same. Thus it is able to ignore the term α^{-m} in (2).

Our task is to consider the integral:

$$\int_W \chi_I \left(-2 \sum_{j=1}^r \cos(2\pi x_j) \right) dx_1 \cdots dx_r.$$

Applying (9) to the interval I , we get the assertion of Theorem 1. $\square$

3. Proof of Theorem 2

Let us follow the proof of Theorem 1 with $r = 1, 2$. In this case, we have

$$\begin{aligned} Y &:= \left| \int_W \left\{ B_K(-z-b) + B_K(z+a) \right\} dx_1 \cdots dx_r \right| \\ &= \frac{2(2\pi)^{-r}}{K+1} \sum_{k=1}^K \left| f\left(\frac{k}{K+1}\right) \right| |\sin \pi k(a-b)| k^{-r/2} + O(K^{-1/2}). \end{aligned} \quad (12)$$

Let ε be a small positive number and take a small positive η such that $\pi\eta/(\sin \pi\eta) < 1 + \varepsilon$ and a large integer K such that $1/(b-a) < \eta(K+1) < K$. We also introduce another parameter $0 < v < 1$ which is chosen later. Divide the summation in (12) into three parts

$$\frac{2(2\pi)^{-r}}{K+1} \left\{ \sum_{k \leq \frac{v}{b-a}} + \sum_{\frac{v}{b-a} < k \leq \eta(K+1)} + \sum_{\eta(K+1) < k \leq K} \right\} =: S_1 + S_2 + S_3.$$

If $b-a \leq v$, using $|\sin \pi k(b-a)| \leq \pi k(b-a)$ and (3), we get

$$S_1 \leq \begin{cases} \frac{(1+\varepsilon)(b-a)}{\pi} \left(2\sqrt{\frac{v}{b-a}} - 1 \right) + O\left(\frac{1}{K}\right) & r = 1, \\ \frac{(1+\varepsilon)(b-a)}{2\pi^2} \left(\log \frac{v}{b-a} + 1 \right) + O\left(\frac{1}{K}\right) & r = 2, \end{cases}$$

while if $b-a > v$, S_1 is trivially zero. If $b-a \leq v$, the trivial bound $|\sin \pi k(b-a)| \leq 1$ implies, for $r = 1, 2$,

$$S_2 \leq \frac{4(1+\varepsilon)}{(2\pi)^{r+1}} \left(\frac{b-a}{v} \right)^{\frac{r}{2}} \left(\frac{2}{r} + \frac{b-a}{v} \right) + O(K^{-1/2}),$$

while if $b-a > v$,

$$S_2 \leq \frac{4(1+\varepsilon)}{(2\pi)^{r+1}} \zeta \left(1 + \frac{r}{2} \right) + O(K^{-1/2}).$$

Finally we have $S_3 = O(K^{-1/2})$ for $r = 1, 2$. The implied constants do not depend on K . Now we let $K \rightarrow \infty$.

In the case $r = 1$ we get

$$Y \leq \begin{cases} \frac{(1+\varepsilon)\sqrt{b-a}}{\pi} \left\{ 2 \left(\sqrt{v} + \frac{1}{\pi\sqrt{v}} \right) - \sqrt{b-a} + \frac{b-a}{\pi v^{\frac{3}{2}}} \right\} & b-a \leq v, \\ \frac{1+\varepsilon}{\pi^2} \zeta\left(\frac{3}{2}\right) & b-a > v. \end{cases}$$

Taking $v = 1/\pi$, it follows that

$$Y \leq 4\pi^{-\frac{3}{2}}(1+\varepsilon)\sqrt{b-a}.$$

For $r = 2$, we have

$$Y \leq \begin{cases} \frac{(1+\varepsilon)(b-a)}{2\pi^2} \left(\log \frac{1}{b-a} + 1 + \frac{1}{\pi v} + \log v + \frac{b-a}{\pi v^2} \right) & b-a \leq v, \\ \frac{1+\varepsilon}{2\pi^3} \zeta(2) & b-a > v. \end{cases}$$

Now taking $v = 1/\sqrt{\pi}$, we get

$$Y \leq \frac{(1+\varepsilon)(b-a)}{2\pi^2} \left(\log \frac{1}{b-a} + 1 + (b-a) \right).$$

The same estimates are valid for

$$\int_W \left\{ B_K(z+b) + B_K(-z-a) \right\} dx_1 \cdots dx_r$$

with $r = 1, 2$. Since ε is chosen arbitrarily, we obtain Theorem 2.

4. Examples

To illustrate the result, we give examples of distributions for Salem numbers of degree 4, 6 and 8. The interval $[0, 1]$ is divided into 100 pieces. We computed the fractional part of α^n for $1 \leq n \leq 200000$, and counted the number of n so that the fractional part of α^n falls into each subintervals. The vertical axis indicates the number of such n .

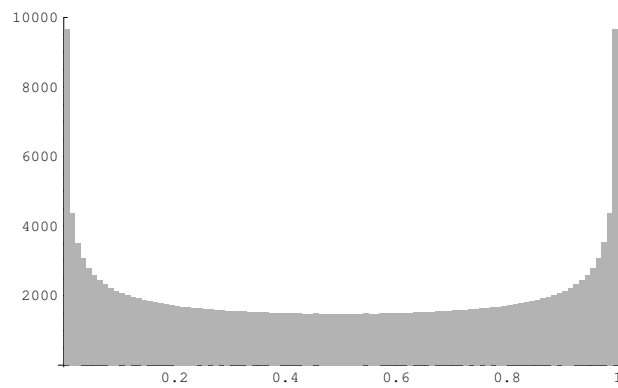


Figure 1. Salem number for $x^4 - x^3 - x^2 - x + 1 = 0$

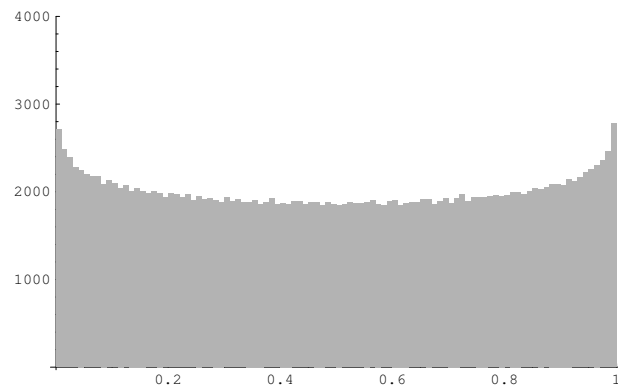


Figure 2. Salem number for $x^6 - x^5 - x^4 + x^3 - x^2 - x + 1 = 0$

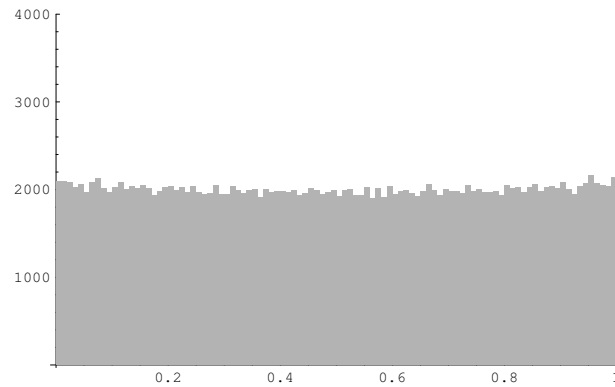


Figure 3. Salem number for $x^8 - 2x^7 + x^6 - x^4 + x^2 - 2x + 1 = 0$

ACKNOWLEDGEMENTS. The authors express their sincere gratitude to the referee for many valuable comments.

References

- [1] M. J. BERTIN, A. DECOMPS-GUILLOUX, M. GRANDET-HUGOT, M. PATHIAUX-DELEFOSSE and J. P. SCHREIBER, Pisot and Salem numbers, *Birkhäuser Verlag, Basel – Boston – Berlin*, 1992.
- [2] F. BEUKERS, Fractional parts of powers of $3/2$, *Prog. Math.* **22** (1982), 13–18.
- [3] D. W. BOYD, Salem numbers of degree four have periodic expansions, *Théorie des nombres, Number Theory, Walter de Gruyter, Berlin, New York*, 1989, 57–64.
- [4] M. DRMOTA and R. F. TICHY, Sequences, Discrepancies and Applications, *Lecture Notes in Math.* 1651, *Springer*, 1997.
- [5] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER and F. G. TRICOMI, Higher Transcendental Functions, Vol. 2, *McGraw-Hill, New York*, 1953.
- [6] L. KUIPERS and H. NIEDERREITER, Uniform distribution of sequences, *Pure and Applied Math.*, *John Wiley & Sons*, 1974.
- [7] M. B. LEVIN, On the complete uniform distribution of the fractional parts of the exponential function, *Trudy Sem. Petrovsk.*, no. 7 (1981), 245–256 (in Russian).

- [8] H. L. MONTGOMERY, Ten lectures on the interface between analytic number theory and harmonic analysis, Conference Board of the Math. Sci., Vol. 84, AMS, 1994.
- [9] G. N. WATSON, A Treatise on the Theory of Bessel Functions, (CML edition), *Cambridge University Press, New York*, 1995.

SHIGEKI AKIYAMA
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
NIIGATA UNIVERSITY
IKARASHI 2-8050, NIIGATA 950-2181
JAPAN

E-mail: akiyama@math.sc.niigata-u.ac.jp

YOSHIO TANIGAWA
GRADUATE SCHOOL OF MATHEMATICS
NAGOYA UNIVERSITY
CHIKUSA-KU, NAGOYA 464-8602
JAPAN

E-mail: tanigawa@math.nagoya-u.ac.jp

(Received November 15, 2002; revised April 14, 2003)