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Integer points on rational curves with fixed gcd

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Abstract. Let $F(X,Y)$ be an irreducible polynomial with integer coefficients of degree ≥ 2 such that the curve C defined by the equation $F(X,Y)=0$ has infinitely many integer points and the point $(0,0)$ is simple on C. In this paper we obtain an explicit upper bound for the size of integer points (x,y) of C for which $\gcd(x,y)$ is bounded.

1. Introduction

Let $F(X,Y)$ be a non-zero polynomial, with integer coefficients, which is irreducible in $\mathbb{Q}[X,Y]$ and satisfies $F(0,0)=0$. In 1929, Skolem [14], [15, page 90] proved, using Runge's theorem about Diophantine equations, that the equation $F(X,Y)=0$ has only finitely many solutions $(x,y)\in\mathbb{Z}^2$ for which $\gcd(x,y)$ is bounded. In 1992, G. Walsh [16, Theorem 2], using an effective version of Runge's theorem, calculated an explicit upper bound for the size of these solutions. The purpose of this paper is to improve the above result, in the case where the equation $F(X,Y)=0$ has infinitely many integer solutions, using a different approach.

Let P be a point of the projective space $\mathbb{P}^r(\mathbb{Q})$ over $\mathbb{Q}$. Choose homogeneous coordinates $(x_0:\ldots:x_r)$ for P so that $x_i\in\mathbb{Z}$ and $\gcd(x_0,\ldots,x_r)=1$. The height of P is defined by

$$H(P)=\max\big\{|x_0|,\ldots,|x_r|\big\}.$$

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For $x \in \mathbb{Q}$ we define $H(x) = H((1 : x))$. Further, if G is a nonzero polynomial with coefficients in $\mathbb{Q}$, we define the height $H(G)$ of G as the height of the point whose coordinates are the coefficients of G (in any order). For an account of the properties of heights, see [3], [1], [13].

Theorem 1. *Let $F(X,Y)$ be an irreducible polynomial in $\mathbb{Z}[X,Y]$ of degree $n \geq 2$ in Y and of degree $m \geq 1$ in X with $n \geq m$. Suppose that the curve C defined by the equation $F(X,Y) = 0$ has infinitely many integer points and the point $(0,0)$ is simple on C. Then, if d is a positive integer and (x,y) an integer solution of $F(X,Y) = 0$ with $\gcd(x,y) = d$, we have*

$$\max\{|x|,|y|\} < d^{2n}(2(mn+1))^{430m^2n^6} H(F)^{230m^2n^6}.$$

G. Walsh communicated me that his method, under the assumptions of the aforementioned theorem, gives the following estimate:

$$\max\big\{|x|,|y|\big\} < (3dmnH(F))^{6m^7n^7}.$$

We remark that the bound of the above theorem is sharper and especially the dependence of d.

Recall that if $F(X,Y)$ is an irreducible polynomial of $\mathbb{Z}[X,Y]$ such that the equation $F(X,Y) = 0$ has infinitely many integer solutions, then [5, page 122] implies that $F(X,Y)$ is an absolutely irreducible polynomial.

Suppose next that $F(X,Y)$ is an absolutely irreducible polynomial of $\mathbb{Z}[X,Y]$ and denote by C the curve defined by the equation $F(X,Y) = 0$. Let $\overline{\mathbb{Q}}$ be an algebraic closure of the field of rational numbers $\mathbb{Q}$ and $\overline{\mathbb{Q}}(C)$ the function field of C over $\overline{\mathbb{Q}}$. Consider the valuation ring V_∞ of $\overline{\mathbb{Q}}(X)$ consisting of all elements $f(X)/g(X)$ such that $\deg f \leq \deg g$. We denote by C_∞ the set of all discrete valuation rings V of $\overline{\mathbb{Q}}(C)$ such that $V \cap \overline{\mathbb{Q}}(X) = V_\infty$. By the classical Siegel's theorem on the integer points, the hypothesis that the equation $F(X,Y) = 0$ has infinitely many integer solutions implies that the curve C is rational with $|C_\infty| \leq 2$.

We call an element V of C_∞ *defined over* a subfield k of $\overline{\mathbb{Q}}$, if $\tau(V) = V$ for every $\tau \in \mathrm{Gal}(\overline{k}/k)$. Furthermore, we say that two elements V and W of C_∞ are *conjugate* over a quadratic field k, if V and W are defined over k and there is $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which is not the identity on k such that $\sigma(V) = W$. In view of Theorem 5.2 of [9], which gives a necessary and sufficient condition for the curve C to have infinitely many integer points, the above statement is equivalent to the following one.

Theorem 2. *Let $F(X,Y)$ be an absolutely irreducible polynomial in $\mathbb{Z}[X,Y]$ of degree $n \geq 2$ in Y and of degree m in X with $n \geq m$ such that the curve C defined by the equation $F(X,Y)=0$ is rational. Suppose that the point $(0,0)$ is simple on C and that $|C_\infty| = 1$ or $|C_\infty| = 2$ and the two elements of C_∞ are conjuguate over a real quadratic field. If d is a positive integer and (x,y) an integer solution of $F(X,Y)=0$ with $\gcd(x,y)=d$, then x and y satisfy the inequality of Theorem 1.*

The paper is organised as follows. In Section 2, an effective version of Riemann–Roch theorem due to W. Schmidt [12] and a version of the implicit functions theorem due to M. Laurent and D. Roy [4] enable us to obtain an effective parametrization of the rational curve C. In Section 3, using this parametization, we prove Theorem 1.

2. Construction of a parametrization

Let $F(X,Y)$ be an absolutely irreducible polynomial in $\mathbb{Z}[X,Y]$. We denote by $\Sigma(C)$ the set of discrete valuation rings W of the function field $\overline{\mathbb{Q}}(C)$ of C such that $\overline{\mathbb{Q}} \subset W$. A divisor D on C is a formal sum

$$D = a_1 W_1 + \cdots + a_s W_s,$$

where $a_1, \ldots, a_s \in \mathbb{Z}$ and $W_1, \ldots, W_s$ are pairwise distinct elements of $\Sigma(C)$. Given $f \in \overline{\mathbb{Q}}(C)$ and $W \in \Sigma(C)$, we denote by $\mathrm{ord}_W(f)$ the order of the function f at W. Let $L(D)$ be the set of functions $f \in \overline{\mathbb{Q}}(C)$ having $\mathrm{ord}_{W_i}(f) \geq -a_i$ and $\mathrm{ord}_W(f) \geq 0$ for every $W \in \Sigma(C)$ with $W \neq W_i$ $(i = 1, \ldots, s)$. Then $L(D)$ is a finite-dimensional vector space over $\overline{\mathbb{Q}}$ (see [2]).

Lemma 1. *Let $F(X,Y)$ be an absolutely irreducible polynomial in $\mathbb{Z}[X,Y]$ of degree $n \geq 2$ in Y and of degree m in X with $n \geq m$. Suppose that the curve C defined by the equation $F(X,Y)=0$ is rational and the point $(0,0)$ is simple on C. Then the curve C admits a parametrization given by*

$$X = \frac{\alpha(T)}{\beta(T)}, \quad Y = \frac{\gamma(T)}{\delta(T)},$$

where $\alpha(T), \beta(T), \gamma(T), \delta(T) \in \mathbb{Z}[T]$ *with* $\deg\alpha, \deg\beta \le n$, $\deg\gamma, \deg\delta \le m$, $\gcd(\alpha,\beta) = \gcd(\gamma,\delta) = 1$ *and*

$$H(\beta(T)X - \alpha(T)) < (2(mn+1))^{32m^2n^4} H(F)^{17m^2n^4},$$

$$H(\delta(T)Y - \gamma(T)) < (2(mn+1))^{42m^3n^3} H(F)^{18m^3n^3}.$$

Proof. Let $P = (0,0)$ and O_P be the local ring of C at P. Since P is a simple point, O_P is a discrete valuation ring. By Riemann–Roch theorem, the space $L(O_P)$ has dimension equal to 2. Theorem A2 of [12] implies that there are polynomials $b_i(X,Y)$ $(i = 1,2)$ and $q(X)$, with

$$\deg_X b_i \le 2mn + 4n - m, \quad \deg_Y b_i \le n-1, \quad \deg q \le mn + n - m,$$

such that the functions f_i represented by the quotients $b_i(X,Y)/q(X)$ on C form a basis of $L(O_P)$. Since the divisor O_P is defined over $\mathbb{Q}$, Theorem B2 of [12] yields that the coefficients of $b_i(X,Y)$ $(i = 1,2)$ and $q(X)$ are integers. If the order of the functions f_j at O_P is > -1, then f_j is a constant. Therefore, the order of f_1 or f_2 at O_P is equal to -1. Suppose that f_1 has this property and put $t = f_1$ and $b(X,Y) = b_1(X,Y)$. Thus we have $\mathbb{Q}(C) = \mathbb{Q}(t)$.

Write $F(X,Y) = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j}X^iY^j$. By Lemmata A1 and A2 of [4], there is only one power series $y(X) = \sum_{s\ge1} c_sX^s$ such that $F(X, y(X)) = 0$ in $\mathbb{Q}[[X]]$. The coefficients of this series satisfy $a_{0,1}^{2s-1}c_s \in \mathbb{Z}$ and for every $v \in M(\mathbb{Q})$ we have

$$|c_s|_v < \max\left\{|a_{i,j}|_v/|a_{0,1}|_v\right\}^{2s-1}(2(m+1)(n+1))^{(2s-1)e(v)} \quad (s \ge 1),$$

where $e(v) = 1$ if the absolute value $|\cdot|_v$ is archimedean and $e(v) = 0$ otherwise. Replacing Lemma 21 in [12] by the above result, we obtain that the vectors $\delta_1, \dots, \delta_n$ of Lemma 26 in [12] satisfy

$$H(\delta_i) < (2(mn+1))^{10m^2n^3+15mn^3} H(F)^{6m^2n^3+9mn^3-3m^2n^2}.$$

Next, the equalities (A.5.6), (B.3.1) in [12] and the above upper bound for $H(\delta_i)$ yield

$$H(b) < (2(mn+1))^{26m^2n^3} H(F)^{15m^2n^3}.$$

Furthermore, [13, Theorem 5.9, p. 211], [12, Lemma 16] and [7, Lemma 4] give

$$H(q) < ((4mn^2H(F))^{2n(mn+n-m)}.$$

Let x and y be the coordinate functions on C. Since $\mathbb{Q}(C) = \mathbb{Q}(t)$, we have that

$$x = \frac{\alpha(t)}{\beta(t)} \quad \text{and} \quad y = \frac{\gamma(t)}{\delta(t)},$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$ are polynomials of $\mathbb{Z}[t]$ with $\gcd(\alpha(t), \beta(t)) = \gcd(\gamma(t), \delta(t)) = 1$. Put $E(X, Y, T) = b(X, Y) - q(X)T$. We may suppose, without loss of generality, that 1 occurs as coefficient of $b(X, Y)$ and $q(X)$. So $H(E) \leq H(b)H(q)$. We denote by $R(X, T)$ the resultant of $E(X, Y, T)$ and $F(X, Y)$ with respect to Y. By [8, Lemma 3.2], we obtain $\deg_T R \leq n$, $\deg_X R \leq 2mn^2 + 4n^2 - m$ and

$$H(R) < (2(mn+1))^{30m^2n^4} H(F)^{17m^2n^4}.$$

The polynomial $\beta(T)X - \alpha(T)$ divides $R(X, T)$. Then $\deg \alpha, \deg \beta \leq n$ and [1, Proposition B.7.3, page 228] implies

$$H(\beta(T)X - \alpha(T)) < (2(mn+1))^{32m^2n^4} H(F)^{17m^2n^4}.$$

Next, we denote by $S(Y, T)$ the resultant of $E(X, Y, T)$ and $F(X, Y)$ with respect to X. By [8, Lemma 3.2], we obtain $\deg_T S \leq m$, $\deg_X S \leq 2mn^2 + 4n^2 - m$ and

$$H(S) < (2(mn+1))^{39m^3n^3} H(F)^{18m^3n^3}.$$

Since $\delta(T)Y - \gamma(T)$ divides $S(X, T)$, it follows that $\deg \gamma, \deg \delta \leq m$ and [1, Proposition B.7.3, page 228] gives

$$H(\delta(T)Y - \gamma(T)) < (2(mn+1))^{42m^3n^3} H(F)^{18m^3n^3}.$$

□

3. Proof of Theorem 1

Since the equation $F(X, Y) = 0$ has infinitely many integer solutions, [11] yields that the highest homogeneous part of $F(X, Y)$ is up to a constant factor a power of a linear or irreducible indefinite quadratic form. Thus the inequality $n \geq m$ gives $\deg F = n$. By Lemma 1, the curve C

admits a parametrization given by

$$X = \frac{\alpha(T)}{\beta(T)}, \quad Y = \frac{\gamma(T)}{\delta(T)},$$

where $\alpha(T), \beta(T), \gamma(T), \delta(T) \in \mathbb{Z}[T]$ with $\deg \alpha, \deg \beta \leq n$, $\deg \gamma, \deg \delta \leq m$, $\gcd(\alpha, \beta) = \gcd(\gamma, \delta) = 1$ and

$$H(\beta(T)X - \alpha(T)) < (2(mn+1))^{32m^2n^4} H(F)^{17m^2n^4},$$
$$H(\delta(T)X - \gamma(T)) < (2(mn+1))^{42m^3n^3} H(F)^{18m^3n^3}.$$

Moreover, we may suppose, without loss of generality, that the polynomials $\beta(T)X - \alpha(T)$ and $\delta(T)Y - \gamma(T)$ have relatively prime coefficients.

Let $\tilde{C}$ be the projective closure of C. Then we have a birational map $\phi : \mathbb{P}^1 \to \tilde{C}$ given by the correspondence

$$(S, T) \to (u(S, T), v(S, T), w(S, T)),$$

where $u(S, T)$, $v(S, T)$, $w(S, T)$ are relatively prime homogeneous polynomials of $\mathbb{Z}[S, T]$ with

$$\frac{\alpha(T)}{\beta(T)} = \frac{u(1, T)}{w(1, T)}, \quad \frac{\gamma(T)}{\delta(T)} = \frac{v(1, T)}{w(1, T)}.$$

Thus [9, Lemma 2.1] implies that ϕ is a birational morphism of $\mathbb{P}^1$ onto C and $\deg u(S, T) = \deg v(S, T) = \deg w(S, T) = n$. Furthermore, since the point $(0, 0)$ is simple on C, it follows that there are $\eta, \theta \in \mathbb{Z}$ with $\gcd(\eta, \theta) = 1$ such that $u(\eta, \theta) = v(\eta, \theta) = 0$ and $w(\eta, \theta) \neq 0$. If $\eta \neq 0$, then $\alpha(1, \theta/\eta) = 0$ and so [7, Lemma 4] implies that

$$\max\{|\eta|, |\theta|\} = H(\theta/\eta) < 2H(\alpha) \leq 2H(\beta(T)X - \alpha(T)).$$

Hence

$$\max\{|\eta|, |\theta|\} < 2(2(mn+1))^{32m^2n^4} H(F)^{17m^2n^4}.$$

Dividing the polynomials $w(S, T)X - u(S, T)$ and $w(S, T)Y - v(S, T)$ by the g.c.d. of their coefficients, we obtain homogeneous polynomials $u_1(S, T)$, $v_1(S, T)$,$w_1(S, T)$ in $\mathbb{Z}[S, T]$ and integers a_1, a_2, such that the polynomials $a_1 w_1(S, T)X - u_1(S, T)$, $a_2 w_1(S, T)Y - u_2(S, T)$ and $w_1(S, T)$

have relatively prime coefficients. The equalities $u(\eta,\theta)=v(\eta,\theta)=0$ imply that there exist homogeneous polynomials $u_2(S,T)$, $v_2(S,T)$ in $\mathbb{Z}[S,T]$ such that

$$\theta^n u_1(S,T)=(\theta S-\eta T)u_2(S,T),\quad \theta^n v_1(S,T)=(\theta S-\eta T)v_2(S,T),$$

if $\theta\neq 0$, and

$$u_1(S,T)=Tu_2(S,T),\quad v_1(S,T)=Tv_2(S,T),$$

otherwise.

Let (x,y) be a simple integer point on C with $\gcd(x,y)=d$. Then $x=dx'$ and $y=dy'$, where $\gcd(x',y')=1$. We have the following two cases:

1. $|C_\infty|=1$. By [9, Lemma 2.2], it follows that $w_1(S,T)=(bS+cT)^n$, where b, c are integers with $\gcd(b,c)=1$. Since $w(\eta,\theta)\neq 0$, we have $(\eta:\theta)\neq(c:-b)$ which is equivalent to $b\eta+c\theta\neq 0$. By [9, Lemma 2.1], there are $s,t\in\mathbb{Z}$ with $\gcd(s,t)=1$ such that

$$x=\frac{u_1(s,t)}{a_1(bs+ct)^n}\quad\text{and}\quad y=\frac{v_1(s,t)}{a_2(bs+ct)^n}.$$

Suppose first that $\theta\neq 0$. We have

$$\theta^n dx' a_1(bs+ct)^n=(\theta s-\eta t)u_2(s,t),$$

$$\theta^n dy' a_2(bs+ct)^n=(\theta s-\eta t)v_2(s,t).$$

Since $\gcd(x',y')=1$, it follows that $\theta s-\eta t$ divides $\theta^n dl(bs+ct)^n$, where l is the l.c.m. of a_1 and a_2. It is easily seen that the g.c.d. of $bs+ct$ and $\theta s-\eta t$ divides $b\eta+c\theta$. Thus we obtain that $\theta s-\eta t$ divides $\theta^n ld(b\eta+c\theta)^n$. On the other hand, using the euclidean algorithm for polynomials, we deduce that there exist two homogeneous polynomials $q_i(S,T)$ $(i=1,2)$ in $\mathbb{Z}[S,T]$ such that

$$b^n u_1(S,T)=(bS+cT)q_1(S,T)+T^n u_1(-c,b),$$

$$(-c)^n u_1(S,T)=(bS+cT)q_2(S,T)+S^n u_1(-c,b).$$

Since x is an integer, we obtain that $bs+ct$ divides $t^n u_1(-c,b)$ and $s^n u_1(-c,b)$, whence it follows that $bs+ct$ divides $u_1(-c,b)$.

Thus we have the following linear system in unknowns s and t:

$$\theta s - \eta t = \kappa, \quad bs + ct = \lambda$$

where κ, λ are integers with κ divides $\theta^n ld(b\eta + c\theta)^n$ and λ divides $u_1(-c, b)$. Hence, we deduce

$$\max\{|s|, |t|\} < \max\{|b|, |c|\}^n \max\{|\eta|, |\theta|\}^{2n-1}(da_1a_2 2^{n-1} + (n+1)|u_1|),$$

where $|u_1|$ denotes the maximum of the absolute values of the coefficients of $u_1(S, T)$. Using the above inequalities, we obtain

$$\max\{|s|, |t|\} < d(2(mn+1))^{81m^2n^5} H(F)^{43m^2n^5}.$$

It follows

$$\max\{|x|, |y|\} < d^n(2(mn+1))^{92m^2n^6} H(F)^{50m^2n^6}.$$

If $\theta = 0$, then we obtain a sharper estimate.

2. $|C_\infty| = 2$. It follows from [9, Lemma 2.2] that $w_1(S, T) = (fS^2 + gST + hT^2)^{n/2}$, where f, g, h are integers with $\gcd(f, g, h) = 1$. By [9, Remark 4.1, p. 485], the disriminant of $fS^2 + gST + hT^2$ is positive and not a perfect square. By [9, Lemma 2.1], there are $s, t \in \mathbb{Z}$ with $\gcd(s, t) = 1$ such that

$$x = \frac{u_1(s,t)}{a_1(fs^2 + gst + ht^2)^{n/2}} \quad \text{and} \quad y = \frac{v_1(s,t)}{a_2(fs^2 + gst + ht^2)^{n/2}}.$$

Suppose first that $\theta \neq 0$. We have

$$\theta^n dx' a_1(fs^2 + gst + ht^2)^{n/2} = (\theta s - \eta t)u_2(s, t),$$

$$\theta^n dy' a_2(fs^2 + gst + ht^2)^{n/2} = (\theta s - \eta t)v_2(s, t).$$

Since $\gcd(x', y') = 1$, it follows that $\theta s - \eta t$ divides $\theta^n dl(fs^2 + gst + ht^2)^{n/2}$, where l is the l.c.m. of a_1 and a_2. The polynomials $fS^2 + gS + h$ and $\theta S - \eta$ are relatively prime. Thus [10, Lemma 3.1] implies that there are $\zeta, \xi \in \mathbb{Z} - \{0\}$ and polynomials $A(S)$, $B(S)$, $\Gamma(T)$, $\Delta(T)$ with integer coefficients, $\deg A, \deg \Gamma \leq 1$, $\deg B, \deg \Delta \leq 2$ and

$$|\zeta|, |\xi| < 2H(f, g, h)^2 H(\eta, \theta)^2$$

such that

$$A(S)(fS^2+gS+h)+B(S)(\theta S-\eta)=\zeta,$$

$$\Gamma(T)(f+gT+hT^2)+\Delta(T)(\theta-\eta T)=\xi.$$

Homogenizing the above equalities, we obtain

$$A_1(S,T)(fS^2+gST+hT^2)+B_1(S,T)(\theta S-\eta T)=\zeta T^{\mu},$$

$$\Gamma_1(S,T)(fS^2+gST+hT^2)+\Delta_1(S,T)(\theta S-\eta T)=\xi S^{\nu},$$

where $A_1(S,T)$, $B_1(S,T)$, $\Gamma_1(S,T)$, $\Delta_1(S,T)$ are homogeneous polynomials of $\mathbb{Z}[S,T]$ with $A_1(S,1)=A(S)$, $B_1(S,1)=B(S)$, $\Gamma_1(1,T)=\Gamma(T)$, $\Delta_1(1,T)=\Delta(T)$, and μ, ν are positive integers ≤ 3. Thus, if $\omega(s,t)=\gcd(fs^2+gst+ht^2,\theta s-\eta t)$, then the above equalities yield that $\omega(s,t)$ divides $\zeta\xi$. Therefore

$$\omega(s,t)<4H(f,g,h)^4H(\eta,\theta)^4<2^6(2(mn+1))^{2^8m^2n^4}H(F)^{136m^2n^4}.$$

The quantity $K=\theta s-\eta t$ divides $\theta^n da_1a_2\omega(s,t)^{n/2}$ and hence

$$|K|<d2^{4n}(2(mn+1))^{192m^2n^5}H(F)^{102m^2n^5}.$$

Since the polynomials $fS^2+gST+hT^2$, $u_1(S,T)$, $v_1(S,T)$ are relatively prime and $fS^2+gST+hT^2$ is irreducible over $\mathbb{Q}$, it follows that at least one of $u_1(S,T)$, $v_1(S,T)$ is not divisible by $fS^2+gST+hT^2$. Denote by $\varrho(S,T)$ this polynomial. We have

$$H((fS^2+gST+hT^2)^{n/2}X-\varrho(S,T))<(2(mn+1))^{42m^2n^4}H(F)^{18m^2n^4}.$$

By [10, Lemma 3.1], there are $R\in\mathbb{Z}-\{0\}$ and polynomials $A(S),B(S)\in\mathbb{Z}[S]$, with $\deg A<n$, $\deg B<n$ and

$$|R|<(2n-1)!H((fS^2+gST+hT^2)^{n/2}X-\varrho(S,T))^{2n-1}$$

such that

$$A(S)(fS^2+gS+h)^{n/2}+B(S)\varrho(S,1)=R.$$

Homogenizating the above equality, we obtain

$$A_1(S,T)(fS^2+gST+hT^2)^{n/2}+B_1(S,T)\varrho(S,T)=RT^{\pi},$$

where $A_1(S,T)$, $B_1(S,T)$ are homogeneous polynomials of $\mathbb{Z}[S,T]$ with $A_1(S,1)=A(S)$, $B_1(S,1)=B(S)$ and π is a positive integer $< 2n$. Since $\Lambda = fs^2+gst+ht^2$ divides $\varrho(s,t)$, the above equality, implies that Λ divides $(Rt^\pi)^{2/n}$. On the other hand, the g.c.d of $fs^2+gst+ht^2$ and t divides f. Hence

$$|\Lambda| < (2(mn+1))^{255m^2n^4} H(F)^{122m^2n^4}.$$

Now, putting $s=(\eta t+K)/\theta$ in the equality $\Lambda = fs^2+gst+ht^2$, we obtain

$$(f\eta^2+g\eta\theta+h\theta^2)t^2+(2\eta Kf+gK\theta)t+fK^2-\Lambda\theta^2=0.$$

Using [6, Corollary 2] and the above inequalities, we obtain

$$|t| < d^2 2^{8n+1}(2(mn+1))^{400m^2n^5} H(F)^{213m^2n^5},$$

$$|s| < d^2 2^{8n+3}(2(mn+1))^{417m^2n^5} H(F)^{222m^2n^5}.$$

Thus, we deduce

$$\max\{|x|,|y|\} < d^{2n}(2(mn+1))^{430m^2n^6} H(F)^{230m^2n^6}.$$

If $\theta=0$, then we obtain a sharper estimate.

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References

[1] M. Hindry and J. Silverman, Diophantine Geometry, *Springer-Verlag*, *New York*, 2000.

[2] S. Lang, Introduction to Algebraic and Abelian Functions, *Springer-Verlag*, 1982.

[3] S. Lang, Diophantine Geometry, *Springer Verlag*, 1983.

[4] M. Laurent and D. Roy, Sur l'approximation algébrique en degré de transcendence un, *Annales de l'Institut Fourier* **49** (1999), 27–55.

[5] E. Maillet, Détermination des points entiers des courbes algeébriques unicursales à coefficients entiers, *Journal de l'Ecole Polytechnique* **2**, 20 (1919), 115–156.

[6] M. Mignotte, An inequality on the greatest roots of a polynomial, *Elem. Math.* **46** (1991), 85–86.

[7] D. Poulakis, Integer Points on Algebraic Curves with Exceptional Units, *J. Austral. Math. Soc. (Series A)* **63** (1997), 145–164.

[8] D. Poulakis, Bounds for the minimal solution of genus zero diophantine equations, *Acta Arithm.* **86/1** (1998), 51–90.

[9] D. Poulakis and E. Voskos, Solving genus zero diophantine equations with at most two infinite valuations, *J. Symbolic Computation* **33** (2002), 479–491.

[10] D. Poulakis, Bounds for the size of integral points on curves of genus zero, *Acta Math. Hungar.* **93**(4) (2001), 327–346.

[11] A. Schinzel, An Improvement of Runge's Theorem on Diophantine Equations, *Pontificia Acad. Sci.* **2** (1969), 1–9.

[12] W. M. Schmidt, Construction and Estimation of Bases in Function Fields, *J. Number Theory* **39**, 2 (1991), 181–224.

[13] J. H. Silverman, The Arithmetic of Elliptic Curves, *Springer Verlag*, 1986.

[14] T. Skolem, Lösung gewisser Gleichungssysteme in ganzen Zahlen oder ganzzahligen Polynomen mit beschränktem gemeinschaftlichen Teiler, *Oslo Vid. Akad. Skr.* **I**, no. 12 (1929).

[15] T. Skolem, Diophantishe Gleichungen, *J. Springer*, *Berlin*, 1938, reprinted by *Chelsea, New York* 1950.

[16] P. G. Walsh, A quantitative version of Runge's theorem on diophantine equations, *Acta Arith.* **63**, no. 2 (1992), 157–172.

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