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Polynomial-exponential equations involving several linear recurrences

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Abstract. Let $\mathcal{E}_A$ denote the ring of complex functions on $\mathbb{N}$ of the form

$$G_n = c_1\alpha_1^n + c_2\alpha_2^n + \cdots + c_t\alpha_t^n,$$

for some $c_i \in \mathbb{C}$ and $\alpha_i \in A$, where $A \subseteq \mathbb{C}$ is multiplicative semigroup. Moreover, let $F(n,y) \in \mathcal{E}_A[y]$. We consider polynomial-exponential Diophantine equations of the form

$$F(n,y) = G_n^{(0)}y^d + G_n^{(1)}y^{d-1} + \ldots + G_n^{(d-1)}y + G_n^{(d)} = 0$$

and show that this equation has only finitely many solutions under certain conditions. This generalizes earlier results due to Corvaja and Zannier (cf. [4], [6]) on such equations.

1. Introduction

Let $A_1, A_2, \ldots, A_k$ and $G_0, G_1, \ldots, G_{k-1}$ be integers and let (G_n) be a k-th order linear recurring sequence given by

$$G_n = A_1G_{n-1} + \cdots + A_kG_{n-k} \quad \text{for } n = k, k+1, \ldots. \tag{1}$$

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Let $\alpha_1, \alpha_2, \ldots, \alpha_t$ be the distinct roots of the corresponding characteristic polynomial

$$X^k - A_1 X^{k-1} - \cdots - A_k. \tag{2}$$

Then for $n \geq 0$

$$G_n = P_1(n)\alpha_1^n + P_2(n)\alpha_2^n + \cdots + P_t(n)\alpha_t^n, \tag{3}$$

where $P_i(n)$ is a polynomial with degree less than the multiplicity of α_i; the coefficients of $P_i(n)$ are elements of the field: $\mathbb{Q}(\alpha_1, \ldots, \alpha_t)$.

We shall be interested in linear recurring sequences (G_n), where all roots of the characteristic polynomial of (G_n) are simple, which means that

$$G_n = c_1\alpha_1^n + c_2\alpha_2^n + \cdots + c_k\alpha_k^n, \tag{4}$$

for some $c_i, \alpha_i \in \mathbb{C}$. If we restrict the roots to come from a multiplicative semigroup $A \subseteq \mathbb{C}$, then we let $\mathcal{E}_A$ denote the ring of complex functions on $\mathbb{N}$ of the form (4) where $\alpha_i \in A$. If $K \subseteq \mathbb{C}$ is a field we define $K\mathcal{E}_A$ by the same formulas, but allowing $c_i \in K$.

Below, A will be usually $\mathbb{Z}$; moreover in that case we define by $\mathcal{E}_{\mathbb{Z}}^+$ the subring formed by those functions having only positive roots, i.e. by the semigroup $\mathbb{N}$. Working in this domain causes no loss of generality: this assumption may be achieved by writing $n = 2m + r$ and considering the cases $r = 0, 1$ separately.

The recurring sequence (G_n) is called nondegenerate, if no quotient α_i/α_j for $1 \leq i < j \leq t$ is equal to a root of unity. Observe that restricting to nondegenerate recurring sequences causes no substantial loss of generality.

In the present paper we deal with Diophantine equations, where linear recurring sequences are involved. Such equations were earlier investigated by several authors, e.g. in the special case

$$G_n = Ex^q, \quad E \in \mathbb{Z}\backslash\{0\}. \tag{5}$$

A survey about this equation can be found in [13, 14] and in more general form in [8, 10].

The first results have been proved just by using elementary and algebraic tools. Later, the results were obtained with the applications of lower bounds for linear forms in logarithms of algebraic numbers.

In 1998, a new development was started by Corvaja and Zannier [4]. They considered linear recurrences defined by

$$G_n = c_1\alpha_1^n + c_2\alpha_2^n + \cdots + c_t\alpha_t^n,$$

where $t \geq 2, c_1, c_2, \ldots, c_t$ are non-zero rational numbers, $\alpha_1 > \alpha_2 > \cdots > \alpha_t > 0$ are integers. They used Schmidt's Subspace Theorem [16], [17] to show that for every integer $q \geq 2$ the equation

$$G_n = x^q \tag{6}$$

has only finitely many solutions $(n, x) \in \mathbb{N}^2$ assuming that G_n is not identically a perfect qth power for all n in a suitable arithmetic progression. Tichy and the first author [10] gave a quantitative version of the above result of Corvaja and Zannier by using a quantitative version of the Subspace Theorem due to Evertse [7].

Recently, Corvaja and Zannier [6] generalized their result. Let K be an algebraic number field and let (G_n) be a nondegenerate linear recurring sequence defined by (4) where $t \geq 2, c_i$ are non-zero elements of K for all $i = 2, \ldots, t$ and where $\alpha_1, \ldots, \alpha_t$ are elements of K with $1 \neq |\alpha_1| > |\alpha_j|$ for all $j = 2, \ldots, t$. Let $f(z, x) \in K[z, x]$ be monic in x and suppose that there do not exist non-zero algebraic numbers d_j, β_j for $j = 1, \ldots, k$ such that

$$f\left(G_n, \sum_{j=1}^{k} d_j\beta_j^n\right) = 0 \tag{7}$$

for all n in an arithmetic progression. Then the number of solutions $(n, x) \in \mathbb{N} \times K$ of the equation

$$f(G_n, x) = 0$$

is finite. The first author [8] gave a quantitative version of this result, which is a little bit more general in the assumptions.

The aim of the present paper is the generalization of the above result to a Diophantine equation where more than one linear recurring sequence is involved. Let $f(x_1, \ldots, x_d, y) \in \overline{\mathbb{Q}}[x_1, \ldots, x_d, y]$ (where $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$) and let $G_n^{(1)}, \ldots, G_n^{(d)} \in \overline{\mathbb{Q}}\mathcal{E}_{\mathbb{Z}}^+$, then we want to consider the Diophantine equation

$$f(G_n^{(1)}, \ldots, G_n^{(d)}, y) = 0.$$

This is equivalent to saying that we consider equations of the form

$$G_n^{(0)} y^d + \cdots + G_n^{(d-1)} y + G_n^{(d)} = 0, \tag{8}$$

where $G_n^{(0)}, \dots, G_n^{(d)} \in \overline{\mathbb{Q}}\mathcal{E}_{\mathbb{Z}}^+$, i.e. to consider polynomials in y with coefficients in the ring $\overline{\mathbb{Q}}\mathcal{E}_{\mathbb{Z}}^+$.

Let us mention that similar types of equations, namely

$$f(x, y, \alpha^x) = 0 \quad \text{and} \quad f(x, y, \alpha^x, \beta^y) = 0,$$

where f is a polynomial with complex coefficients and α, β are non-zero complex numbers, were studied by Schmidt [18] and Ahlgren [1], [2]. They showed that this equations can have solutions with arbitrarily large values of $|x|$ only in the case when f and α, β are of a particularly simple form.

The second author [20] showed another result which is related to what we consider here. Let $f(x, y) \in \overline{\mathbb{Q}}[x, y]$ be monic in y, absolutely irreducible and of degree $d \geq 2$ in y; let $g(x) \in \mathbb{Z}[x]$ be a non constant polynomial; let $G_n \in \overline{\mathbb{Q}}\mathcal{E}_{\mathbb{Z}}$ not constant. Then the equation

$$f(G_n, y) = g(n)$$

has only finitely many solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$.

2. Results

In this section we will state our general results. First we need some notation. Let $d \geq 2$ be an integer and let $G_n^{(0)}, \dots, G_n^{(d)} \in \overline{\mathbb{Q}}\mathcal{E}_{\mathbb{Z}}^+$, i.e. we have

$$\begin{aligned} G_n^{(0)} &= a_1^{(0)} \alpha_1^{(0)^n} + a_2^{(0)} \alpha_2^{(0)^n} + \cdots + a_{t^{(0)}}^{(0)} \alpha_{t^{(0)}}^{(0)^n}, \\ &\vdots \\ G_n^{(d)} &= a_1^{(d)} \alpha_1^{(d)^n} + a_2^{(d)} \alpha_2^{(d)^n} + \cdots + a_{t^{(d)}}^{(d)} \alpha_{t^{(d)}}^{(d)^n}, \end{aligned}$$

where $a_i^{(j)}$ are algebraic and $\alpha_i^{(j)}$ are positive integers such that $\alpha_1^{(j)} > \alpha_2^{(j)} > \cdots > \alpha_{t^{(j)}}^{(j)}$ for all $i = 1, \dots t^{(j)}$ and $j = 0, \dots, d$. Now we consider the Diophantine equation

$$G_n^{(0)} y^d + \cdots + G_n^{(d-1)} y + G_n^{(d)} = 0.$$

Let $f(x_0,\dots,x_d,y)=x_0y^d+\cdots+x_{d-1}y+x_d$ be fixed for the rest of the paper. So the above equation becomes

$$f(G_n^{(0)},\dots,G_n^{(d)},y)=0.$$

We will show how to this equation another equation in some normal form can be associated. First, we set (for a positive real determination of the roots)

$$\alpha := \max\left\{ {\alpha_1^{(d)}}^{\frac{1}{d}}, \left(\frac{\alpha_1^{(d-1)}}{{\alpha_1^{(0)}}^{\frac{1}{d}}}\right)^{\frac{1}{d-1}}, \left(\frac{\alpha_1^{(d-2)}}{{\alpha_1^{(0)}}^{\frac{2}{d}}}\right)^{\frac{1}{d-2}}, \dots, \frac{\alpha_1^{(1)}}{{\alpha_1^{(0)}}^{\frac{d-1}{d}}} \right\}$$

$$= \max_{i=1,\dots,d}\left(\frac{\alpha_1^{(i)}}{{\alpha_1^{(0)}}^{\frac{d-i}{d}}}\right)^{\frac{1}{i}}.$$

Moreover, let

$$y=\frac{\alpha^n}{{\alpha_1^{(0)}}^{\frac{n}{d}}}z.$$

Then consider

$$\frac{1}{\alpha^{dn}}f\left(G_n^{(0)},\dots,G_n^{(d)},\frac{\alpha^n}{{\alpha_1^{(0)}}^{\frac{n}{d}}}z\right). \tag{9}$$

This is a polynomial in z with coefficients in $\overline{\mathbb{Q}}\mathcal{E}_A$, where A is the multiplicative group generated by

$$\alpha, {\alpha_1^{(0)}}^{\frac{1}{d}} \text{ and the roots of } G_n^{(0)},\dots,G_n^{(d)},$$

i.e. the coefficients of this polynomial are again power sums. Observe that all the roots which appear in these power sums are ≤ 1, because of our construction and that one of the roots which appear as coefficient of z^d is 1 (this will be proved in Lemma 2 below). Let $\gamma_1,\dots,\gamma_r$ denote the different roots of these power sums (the coefficients of (9) as a polynomial in z), which are strictly less than 1. We identify the expressions γ_i^n in (9) by a new variable x_i. Therefore we get a polynomial (linear in $x_1,\dots,x_r$) $g_f(x_1,\dots,x_r,z)\in\overline{\mathbb{Q}}[x_1,\dots,x_r,z]$ such that

$$g_f(\gamma_1^n,\dots,\gamma_r^n,z)=\frac{1}{\alpha^{dn}}f\left(G_n^{(0)},\dots,G_n^{(d)},\frac{\alpha^n}{{\alpha_1^{(0)}}^{\frac{n}{d}}}z\right).$$

This polynomial is some kind of normal form for our equation under consideration. We denote by $D_{g_f,z}(x_1,\dots,x_r)$ the discriminant of g_f with respect to z and we set $D_{g_f,z} := D_{g_f,z}(0,\dots,0)$.

We are now in the position to formulate our main result. Our aim is to prove that the equation has only finitely many solutions in integers, apart from "trivial" cases which can be classified. We achieve this goal under a suitable technical hypothesis in the main theorem below.

Theorem 1. *Let $d \geq 2$ and let $G_n^{(0)},\dots,G_n^{(d)} \in \overline{\mathbb{Q}}\mathcal{E}_{\mathbb{Z}}^{+}$. Assume that*

$$D_{g_f,z} \neq 0. \tag{10}$$

Then there exist finitely many recurrences $H_n^{(1)},\dots,H_n^{(s)}$ with algebraic coefficients and algebraic roots, arithmetic progressions $\mathcal{P}_1,\dots,\mathcal{P}_s$, and a finite set $\mathcal{N}$ of integers, such that for the set S of solutions $(n,y) \in \mathbb{N}\times\mathbb{Z}$ of the equation

$$f(G_n^{(0)},\dots,G_n^{(d)},y) = G_n^{(0)}y^d + \cdots + G_n^{(d-1)}y + G_n^{(d)} = 0$$

we have

$$S = \bigcup_{i=1}^{s} \big\{(n,H_n^{(i)}) : n \in \mathcal{P}_i\big\} \cup \big\{(n,y) : n \in \mathcal{N},\ y \in \mathbb{Z}\big\} \cup M,$$

where M is a finite set.

Remark 1. In contrast to [6], [8], where Puiseux series are used, the main tool in our proof is the fact that we can use some suitable version of the Implicit Function Theorem to express z in g_f as a power sum of the other variables which converges locally around the origin. In the version of Puiseux's theorem for several variables (cf. [3]) known to the authors, a similar condition to (10) appears, namely that the discriminant does not vanish in a certain region.

Remark 2. As we use the Implicit Function Theorem as our driving tool, it is clear that condition (10) is equivalent to

$$(10) \iff g_f(0,\dots,0,z) \text{ has only simple roots}$$

$$\iff \frac{\partial g_f}{\partial z}(0,\dots,0,z_i) \neq 0,\ i = 1,\dots,d,$$

where z_i are the roots of $g_f(0,\dots,0,z)$. Note that a similar condition already appeared in the main result in [5].

Let us point out what the construction of α means for the equation

$$G_n = a_1\alpha_1^n + \cdots + a_t\alpha_t^n = y^d.$$

We get

$$g_f(x_1,\dots,x_{t-1},z) = z^d - (a_1 + a_2x_1 + \cdots + a_tx_{t-1}),$$

such that

$$g_f\left(\frac{\alpha_2^n}{\alpha_1^n},\dots,\frac{\alpha_t^n}{\alpha_1^n},z\right) = 0.$$

Observe that condition (10) is satisfied and therefore we get once again the result of Corvaja and Zannier [4]. In fact the construction is exactly the same as the idea of their proof.

Remark 3. Note that the progressions can be chosen in such a way that along the progression $\mathcal{P}_i$ we have

$$H_n^{(i)} \in \mathcal{E}_{\mathbb{Z}}^+.$$

We will show this at the end of the proof of Theorem 1.

Remark 4. Observe that the first type of infinite families of solutions may appear. For example the equation

$$y^2 + (2^n + 3^n)y + 6^n = 0$$

has solutions $(n, -2^n)$ and $(n, -3^n)$ for all $n \in \mathbb{N}$. Moreover, observe that

$$g_f(x_1, x_2, z) = z^2 + (1 + x_1)z + x_1$$

and therefore

$$D_{g_f,z} = 1 \neq 0.$$

These solutions are "trivial" infinite families of solutions in the sense that the equation has already solutions in the ring of power sums (cf. also the results due to Corvaja and Zannier [4, 6]).

Remark 5. In fact it is easy to decide whether such infinite families of solutions appear or not. This follows from the fact that the ring $\mathcal{E}_A$, where

A is the multiplicative group generated by the roots of a recurrence G_n, is easily understood. It is well known (see [15]) that this ring is isomorphic to the ring

$$\mathbb{C}[T_1, \dots, T_t, T_1^{-1}, \dots, T_t^{-1}]$$

if A has rank $t \geq 1$ and has no torsion. We simply choose a basis $\gamma_1, \dots, \gamma_t$ of A and associate the variable T_i to the function $n \mapsto \gamma_i^n$.

Remark 6. It is easy to see that some further infinite families can appear. For example consider the equation

$$(2^n - 2)y^2 + 3^n - 3 = 0.$$

Obviously, this equation has infinitely many solutions, namely $(1, y)$ with $y \in \mathbb{Z}$ arbitrary. We have

$$g_f(x_1, x_2, z) = (1 - 2x_1)z^2 + 1 - 3x_2,$$

therefore $D_{g_f,z} = -4 \neq 0$. Let us notice that the assumption that the polynomial f is monic excludes the existence of "trivial" families of solutions of this type.

We state some corollaries, which follow from the proof of our main theorem.

Corollary 1. *Let $d \geq 2$ and let $G_n^{(1)}, \dots, G_n^{(d)} \in \overline{\mathbb{Q}}\mathcal{E}_{\mathbb{Z}}^+$. Assume that there does not exist $H_n \in \overline{\mathbb{Q}}\mathcal{E}_{\overline{\mathbb{Q}}}$ such that*

$$H_n^d + G_n^{(1)} H_n^{d-1} + \cdots + G_n^{(d)} = 0$$

for all n in a certain arithmetic progression. Moreover, assume that

$$\alpha_1^{(d-1)} > \alpha_1^{(j)^{d-1}} \quad \text{for all } j = 0, \dots, d-2, d. \tag{11}$$

Then the Diophantine equation

$$y^d + G_n^{(1)} y^{d-1} + \cdots + G_n^{(d-1)} y + G_n^{(d)} = 0$$

has only finitely many solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$.

The next corollaries handle equations of the form under consideration of degree 3.

Corollary 2. *Let $G_n^{(1)}, G_n^{(2)} \in \overline{\mathbb{Q}}\mathcal{E}_{\mathbb{Z}}^{+}$. Assume that there does not exist $H_n \in \overline{\mathbb{Q}}\mathcal{E}_{\overline{\mathbb{Q}}}$ such that*

$$H_n^3 + G_n^{(1)} H_n + G_n^{(2)} = 0$$

for all n in a certain arithmetic progression. Moreover, assume that

$$\big(\alpha_1^{(1)}\big)^3 \neq \big(\alpha_1^{(2)}\big)^2 \quad \text{or} \quad 4\big(a_1^{(1)}\big)^3 + 27\big(a_2^{(2)}\big)^2 \neq 0.$$

Then the equation

$$y^3 + G_n^{(1)} y + G_n^{(2)} = 0$$

has only finitely many solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$.

Corollary 3. *Let $0 \neq a, b \in \overline{\mathbb{Q}}$ and α, β positive integers. Then the equation*

$$y^3 + a\alpha^n y + b\beta^n = 0$$

has for

$\alpha^3 \neq \beta^2$ *only finitely many solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$, and for*

$\alpha^3 = \beta^2$ *the set of solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$ is contained in the following set consisting of infinite families*

$$\{(n, c\beta^{\frac{n}{3}}) : n \in \mathbb{N},\ c^3 + ac + b = 0\}.$$

Although we work only in $\mathcal{E}_{\mathbb{Z}}$, similar results should hold more generally for functions with algebraic roots, with the restriction that we have dominating roots and the recurrences are nondegenerate. Also, the results should hold by allowing the coefficients to be polynomials in n with the restriction that the coefficient of the dominating root is constant.

Moreover, it is clear that the results can be quantified by using quantitative versions of the Subspace Theorem, e.g. due to Evertse [7].

3. Auxiliary results

The proof of our theorem depends on a version of the Subspace Theorem due to Schlickewei (the following version can be found in [17], [16]).

Let K be an algebraic number field. Denote its ring of integers by O_K and its collection of places by M_K. For $v \in M_K$, $x \in K$, we define the

absolute value $|x|_v$ by

(i) $|x|_v = |\sigma(x)|^{1/[K:\mathbb{Q}]}$ if v corresponds to the embedding $\sigma : K \hookrightarrow \mathbb{R}$;

(ii) $|x|_v = |\sigma(x)|^{2/[K:\mathbb{Q}]} = |\bar{\sigma}(x)|^{2/[K:\mathbb{Q}]}$ if v corresponds to the pair of conjugate complex embeddings $\sigma, \bar{\sigma} : K \hookrightarrow \mathbb{C}$;

(iii) $|x|_v = (N\wp)^{-\mathrm{ord}_\wp(x)/[K:\mathbb{Q}]}$ if v corresponds to the prime ideal $\wp$ of O_K.

Here $N\wp = \#(O_K/\wp)$ is the norm of $\wp$ and $\mathrm{ord}_\wp(x)$ the exponent of $\wp$ in the prime ideal decomposition of (x), with $\mathrm{ord}_\wp(0) := \infty$. In case (i) or (ii) we call v real infinite or complex infinite, respectively; in case (iii) we call v finite. These absolute values satisfy the *Product formula*

$$\prod_{v \in M_K} |x|_v = 1 \quad \text{for } x \in K\backslash\{0\}. \tag{12}$$

We define the *K-height* of $x \in K$ to be

$$\mathcal{H}_K(x) = \prod_{v \in M_K} \max\{1, |x|_v\}.$$

Observe that $\mathcal{H}_{\mathbb{Q}}(x) = |x|$ (the usual absolute value) for $x \in \mathbb{Z}$ and that

$$\mathcal{H}_L(x) = \mathcal{H}_K(x)^{[L:K]},$$

for $x \in K$ and for a finite extension L of K.

The *height* of $\mathbf{x} = (x_1, \dots, x_n) \in K^n$ with $\mathbf{x} \neq \mathbf{0}$ is defined as follows: for $v \in M_K$ put

$$|\mathbf{x}|_v = \max_{1 \le i \le n} |x_i|_v.$$

Now define

$$\mathcal{H}(\mathbf{x}) = \prod_{v \in M_K} \max\{1, |\mathbf{x}|_v\}.$$

Theorem 2 (Subspace Theorem, Schlickewei)**.** *Let K be an algebraic number field and let $S \subset M_K$ be a finite set of absolute values which contains all the infinite ones. For $v \in S$, let $L_{1,v}, \dots, L_{n,v}$ be n linearly independent linear forms in n variables with coefficients in K. Let $\delta > 0$ be given. Then the solutions of the inequality*

$$\prod_{v \in S} \prod_{i=1}^{n} |L_{i,v}(\mathbf{x})|_v < \mathcal{H}(\mathbf{x})^{-\delta} \tag{13}$$

with $\mathbf{x} \in (O_K)^n$ and $\mathbf{x} \neq \mathbf{0}$ lie in finitely many proper subspaces of K^n.

This follows from Theorem 1D' in [17], page 178.

Below we have collected two simple lemmas which are needed in our proofs. Namely, we need an estimate for the number of zeros occurring in a linear recurring sequence (this number is called the zero multiplicity of the recurrence). The proof is easy and therefore we give it (cf. [9]).

Lemma 1. *Let (G_n) be a linear recurring sequence defined by $G_n = c_1\alpha_1^n + c_2\alpha_2^n + \cdots + c_t\alpha_t^n$ where $t \geq 1$, c_i are non-zero complex and $\alpha_1 > \cdots > \alpha_t > 0$ are real numbers. Then the number of solutions of the equation*

$$G_n = 0$$

is at most $t - 1$.

Proof. We prove our assertion by induction on t. The case $t = 1$ is trivial. Now consider the function of one real variable

$$g(x) = c_1 \exp(x \log(\alpha_1/\alpha_t)) + \cdots + c_{t-1} \exp(x \log(\alpha_{t-1}/\alpha_t)) + c_t.$$

Observe that by separating real and imaginary parts we can assume that the c_i are real. Clearly, the zeros of g at positive integral points are exactly the zeros of G_n. Now, $g(x)$ is a differentiable function of the real variable x. So, between any two zeros of g one can find a zero of the derivative g' of g. Since the derivative is a function of the same type, with $t - 1$ terms, the inductive hypothesis can be applied and the desired conclusion follows. □

Let us mention the remarkable result that there exists an upper bound (which does only depend on the order t, but in fact triply exponentially) for the zero multiplicity of arbitrary nondegenerate linear recurring sequences of complex numbers due to W. M. Schmidt [19].

As a second lemma we prove that the construction leading to the polynomial g_f has the properties we have claimed.

Lemma 2. *Let $f, G_n^{(0)}, \ldots, G_n^{(d)}$ and α be as at the beginning of Section 2. Then the dominant root in*

$$f\left(G_n^{(0)}, \ldots, G_n^{(d)}, \frac{\alpha^n}{{\alpha_1^{(0)}}^{\frac{n}{d}}} z\right)$$

is α^d and it appears as the coefficient of z^d.

PROOF. Let us assume that

$$\alpha = \frac{{\alpha_1^{(k)}}^{\frac{1}{k}}}{{\alpha_1^{(0)}}^{\frac{d-k}{dk}}}.$$

Thus, we have

$$\alpha \geq \frac{{\alpha_1^{(i)}}^{\frac{1}{i}}}{{\alpha_1^{(0)}}^{\frac{d-i}{di}}} \quad \text{for all } i = 1, \dots, d. \tag{14}$$

It is clear that it is enough to investigate how the dominant roots transform under the substitution

$$y = \frac{\alpha^n}{{\alpha_1^{(0)}}^{\frac{n}{d}}} z.$$

The dominant root in the coefficient of z^d is

$$\alpha_1^{(0)} \frac{\alpha^d}{\alpha_1^{(0)}} = \frac{{\alpha_1^{(k)}}^{\frac{d}{k}}}{{\alpha_1^{(0)}}^{\frac{d-k}{k}}}.$$

Moreover, the dominant root in the coefficient of z^{d-i} for $i = 1, \dots, d$ is

$$\alpha_1^{(i)} \frac{\alpha^{d-i}}{{\alpha_1^{(0)}}^{\frac{d-i}{d}}} = \frac{\alpha_1^{(i)}}{{\alpha_1^{(0)}}^{\frac{d-i}{d}}} \frac{{\alpha_1^{(k)}}^{\frac{d-i}{k}}}{{\alpha_1^{(0)}}^{\frac{(d-k)(d-i)}{dk}}} \leq \frac{{\alpha_1^{(k)}}^{\frac{i}{k}}}{{\alpha_1^{(0)}}^{\frac{(d-k)i}{dk}}} \frac{{\alpha_1^{(k)}}^{\frac{d-i}{k}}}{{\alpha_1^{(0)}}^{\frac{(d-k)(d-i)}{dk}}} = \frac{{\alpha_1^{(k)}}^{\frac{d}{k}}}{{\alpha_1^{(0)}}^{\frac{d-k}{k}}},$$

where we have used the definition of α (especially (14)) and the upper bound is the dominant root of the coefficient of z^d. Observe that for the coefficient of z^{d-k} we get

$$\alpha_1^{(k)} \frac{\alpha^{d-k}}{{\alpha_1^{(0)}}^{\frac{d-k}{d}}} = \frac{{\alpha_1^{(k)}}^{\frac{k}{k}}}{{\alpha_1^{(0)}}^{\frac{d-k}{d}}} \frac{{\alpha_1^{(k)}}^{\frac{d-k}{k}}}{{\alpha_1^{(0)}}^{\frac{(d-k)(d-k)}{dk}}} = \frac{{\alpha_1^{(k)}}^{\frac{d}{k}}}{{\alpha_1^{(0)}}^{\frac{d-k}{k}}},$$

and therefore the dominant root also appears in the power sum which appears as coefficient of z^{d-k}. □

Our last tool is the Implicit Function Theorem. The basic form of the Implicit Function Theorem is the assertion that a function in n variables, of

sufficient smoothness, satisfying an appropriate nondegeneracy condition, can be used to define one of the variables as a function of the other $n-1$ variables. Here we will consider the implicit function theorem in the real analytic category (see [11], page 35 and [12]). We will use the following notation: a *multiindex* α is an element of $\mathbb{N}^m$. Set

$$|\alpha| = |\alpha_1 + \cdots + \alpha_m|.$$

We will write 0 to mean the multiindex $(0, \ldots, 0)$.

Theorem 3 (Implicit Function Theorem)**.** *Suppose the power series*

$$F(x_1, \ldots, x_r, y) = \sum_{|\alpha| \geq 0, k \geq 0} a_{\alpha,k} x_1^{\alpha_1} \cdots x_r^{\alpha_r} y^k$$

is absolutely convergent for $|x_1| + \cdots + |x_r| \leq R_1$, $|y| \leq R_2$. *If*

$$a_{0,0} = 0 \quad \textit{and} \quad a_{0,1} \neq 0$$

then there exist $r_0 > 0$ *and a power series*

$$f(x_1, \ldots, x_r) = \sum_{|\alpha| > 0} c_\alpha x_1^{\alpha_1} \cdots x_r^{\alpha_r} \tag{15}$$

such that (15) is absolutely convergent for $|x_1| + \cdots + |x_r| \leq r_0$ *and*

$$F(x_1, \ldots, x_r, f(x_1, \ldots, x_r)) = 0.$$

Moreover: if the coefficients of F *are algebraic, then the coefficients of* f *are also algebraic.*

Observe that it is clear that the same holds in a more general form. Suppose that $F(x_1, \ldots, x_r, y)$ converges absolutely for $|x_1| + \cdots + |x_r| \leq R_1$ and that $|y - y_0| \leq R_2$ for some $y_0 \in \overline{\mathbb{Q}}$ with $F(0, \ldots, 0, y_0) = 0$. Then under the assumption that

$$\frac{\partial F}{\partial y}(0, \ldots, 0, y_0) \neq 0.$$

the conclusion is that there exists a power series

$$f(x_1, \ldots, x_r) = \sum_{|\alpha| \geq 0} c_\alpha x_1^{\alpha_1} \cdots x_r^{\alpha_r}$$

for which the same as above holds.

4. Proof of the main theorem

In the sequel $C_1, C_2, \dots$ will denote positive numbers depending only on the coefficients and roots of $G_n^{(0)}, \dots, G_n^{(d)}$.

According to Lemma 1 the number of solutions of our equation

$$G_n^{(0)} y^d + \cdots + G_n^{(d)} = 0$$

of the form $(n, 0)$, $n \in \mathbb{N}$ can be estimated by $t^{(d)}$. Moreover, the set $\mathcal{N}$ of solutions of

$$G_n^{(0)} = G_n^{(1)} = \cdots = G_n^{(d)} = 0$$

leads trivially to finitely many families of solutions of the form (n, y), with $n \in \mathcal{N}$, $y \in \mathbb{Z}$. Consequently, we can restrict ourselves to solutions of the form $(n, y) \in \mathbb{N} \times \mathbb{Z}$ with $n \notin \mathcal{N}$ and $y \neq 0$, i.e. to solutions $(n, z) \in (\mathbb{N} \backslash \mathcal{N}) \times K$ of

$$g_f(\gamma_1^n, \dots, \gamma_r^n, z) = 0 \tag{16}$$

with $z \neq 0$, where K is the number field generated by the coefficients of $G_n^{(0)}, \dots, G_n^{(d)}$, and by

$$\alpha, \alpha_1^{(0)^{\frac{1}{d}}}.$$

From now on we will look at a subsequence of such solutions, which we will denote by $(n, y_n) \in \mathbb{N} \times \mathbb{Z}$ (respectively $(n, z_n) \in \mathbb{N} \times K$) with $n \in \Sigma$, where Σ is a set of positive integers.

First, we show that the sequence $(z_n)_{n \in \Sigma}$ must be bounded. Assume that this is not the case. We can write (see Lemma 2)

$$g_f(x_1, \dots, x_r, z) = (1 + p_0(x_1, \dots, x_r)) z^d + \cdots + p_d(x_1, \dots, x_r)$$

with polynomials $p_i(x_1, \dots, x_r)$ for which $p_0(0, \dots, 0) = 0$ holds. Dividing by z_n^{d-1}, we get

$$\begin{aligned}(1 + p_0(\gamma_1^n, \dots, \gamma_r^n)) z_n \\ = -p_1(\gamma_1^n, \dots, \gamma_r^n) - p_2(\gamma_1^n, \dots, \gamma_r^n) z_n^{-1} - \cdots - p_d(\gamma_1^n, \dots, \gamma_r^n) z_n^{-d+1}.\end{aligned}$$

From this it follows that there exist constants d_1, d_2 such that we have

$$d_1 \leq |(1 + p_0(\gamma_1^n, \dots, \gamma_r^n)) z_n| \leq d_2.$$

Since

$$1 + p_0(\gamma_1^n, \ldots, \gamma_r^n) \longrightarrow 1, \quad \text{for } n \longrightarrow \infty,$$

we conclude that z_n is bounded, which is a contradiction.

From this discussion it follows that all solutions (n, z_n) lie in the union of arbitrarily small neighborhoods of the solutions of

$$g_f(0, \ldots, 0, z) = 0,$$

at least if n is large enough. In what follows we will only consider those n, lying in a subsequence $\mathcal{R} \subseteq \Sigma$, for which z_n converges.

Now by (10), Remark 2 and the Implicit Function Theorem 3 we can conclude that

$$z = z_0 + \sum_{|i|>0} a_i x_1^{i_1} \ldots x_r^{i_r}$$

with $a_i \in \overline{\mathbb{Q}}$, where z_0 satisfies $g_f(0, \ldots, 0, z_0) = 0$. This series converges around the point $(x_1, \ldots, x_r) = (0, \ldots, 0)$. Therefore for each solution (n, z_n) of (16) with $n \in \mathcal{R}$ sufficiently large, we get

$$z_n = z_0 + \sum_{|i|>0} a_i \gamma_1^{i_1 n} \ldots \gamma_r^{i_r n},$$

for some z_0 and coefficients a_i, and if n is large enough.

Next we are going to approximate z_n by a finite sum extracted from the above expansion. We define

$$V_n := z_0 + \sum_{0<|i|<H} a_i \gamma_1^{i_1 n} \ldots \gamma_r^{i_r n},$$

where $H \geq 1$ is an integer to be chosen later. We may write

$$V_n = z_0 + \sum_{j=1}^{h-1} c_j \beta_j^n, \quad n \in \mathcal{R},$$

where $c_j \in \overline{\mathbb{Q}}$ and the β_j are distinct, less than 1, and lie in the multiplicative group A. Clearly V_n is nondegenerate. Moreover, we have

$$h \leq H^r.$$

We enlarge K at once and assume that it contains all the coefficients c_j.

Observe that we can approximate the error we make by approximating z_n by V_n. We have

$$|z_n - V_n| \le C_1 \max\{\gamma_1, \dots, \gamma_r\}^{Hn} = C_1 C_2^{Hn},$$

because we have

$$z - z_0 - \sum_{0<|i|<H} a_i x_1^{i_1} \dots x_r^{i_r} = \mathcal{O}\left(\max\{x_1, \dots, x_r\}^H\right).$$

We want to point out that $C_2 := \max\{\gamma_1, \dots, \gamma_r\} < 1$.

For later purpose we need an estimate of the K-height of z_n. We derive first an estimate for $\mathcal{H}_K(y_n)$. Observe that we have

$$|y_n| \le C_3 \frac{\alpha^n}{{\alpha_1^{(0)}}^{\frac{n}{d}}},$$

since $|z_n|$ is bounded. Therefore, we get

$$\mathcal{H}_K(y_n) = \mathcal{H}_{\mathbb{Q}}(y_n)^{[K:\mathbb{Q}]} = |y_n|^{[K:\mathbb{Q}]} \le C_4 C_5^n.$$

By our relation

$$z_n = \frac{{\alpha_1^{(0)}}^{\frac{n}{d}}}{\alpha^n} y_n$$

and the fact that the height of a product can be bounded by the product of the heights we conclude

$$\mathcal{H}_K(z_n) \le C_4 \mathcal{H}_K\left(\frac{{\alpha_1^{(0)}}^{\frac{1}{d}}}{\alpha}\right)^n C_5^n = C_6 C_7^n.$$

Observe that we need here that $y_n \in \mathbb{Z}$.

We choose H so that

$$C_2^H C_7 < 1. \tag{17}$$

From now on H is fixed and therefore also h, c_i, β_i for $i = 1, \dots, h$ are fixed. We choose a finite set S of absolute values of K so that it contains all infinite absolute values. Moreover we require that all $\alpha_i^{(j)}$, $i = 1, \dots, t^{(j)}$,

$j = 0, \dots, d$ are S-units. In particular, with this choice all the β_j are S-units. Also, the z_n are S-units, in view of the relation

$$y_n = \frac{\alpha^n}{\alpha_1^{(0)\frac{n}{d}}} z_n$$

and the fact that the y_n are integers.

We shall apply the Subspace Theorem 2, so let us define, for every $v \in S$, $h+1$ independent linear forms in $\mathbf{X} := (X_0, \dots, X_h)$ as follows: put

$$L_{0,\infty}(\mathbf{X}) = -X_0 + X_1 + c_1 X_2 \cdots + c_{h-1} X_h,$$

and for $v \in S$, $0 \le i \le h$, $(i, v) \ne (0, \infty)$ put

$$L_{i,v}(\mathbf{X}) = X_i.$$

Here ∞ denotes the infinite absolute value, which coincides with the complex absolute value in the embedding of K in $\mathbb{C}$. For $n \in \mathcal{R}$ define the vectors

$$\mathbf{x}_n = (-z_n, 1, \beta_1^n, \dots, \beta_{h-1}^n) \in (O_K)^{h+1}$$

and consider the double product

$$\prod_{v \in S} \prod_{i=0}^{h} |L_{i,v}(\mathbf{x}_n)|_v.$$

By putting

$$\sigma = -z_n + z_0 + c_1 \beta_1^n + \cdots + c_{h-1} \beta_{h-1}^n = L_{0,\infty}(\mathbf{x}_n),$$

we can rewrite the double product as

$$|\sigma|_\infty \left(\prod_{v \in S \setminus \{\infty\}} |-z_n|_v \right) \left(\prod_{v \in S} \prod_{i=1}^{h-1} |\beta_i|_v \right).$$

Observe that the β_i are S-units for $i \ge 1$. In particular, this implies by the Product formula (12)

$$\left(\prod_{v \in S} \prod_{i=1}^{h-1} |\beta_i^n|_v \right) = 1.$$

Moreover, we get

$$\left(\prod_{v\in S\setminus\{\infty\}} |-z_n|_v\right) \le \mathcal{H}_K(z_n) \le C_6 C_7^n.$$

Therefore we have

$$\prod_{v\in S}\prod_{i=0}^{h} |L_{i,v}(\mathbf{x}_n)|_v \le C_1 C_6 \left(C_2^H C_7\right)^n,$$

where we have used the bound for the approximation error.

Last we need an upper bound for $\mathcal{H}(\mathbf{x}_n)$. We have

$$\mathcal{H}(\mathbf{x}_n) \le \mathcal{H}_K(z_n)\mathcal{H}_K(\beta_1)^n \dots \mathcal{H}_K(\beta_{h-1})^n \le C_8 C_9^n. \tag{18}$$

Note that the constants do not depend on n.

We now choose δ so that

$$C_2^H C_7 C_9^\delta < 1. \tag{19}$$

This will be possible for small δ in view of (17).

In view of the bound for the double product we derived and (18), the verification of (13) of the Subspace Theorem 2 will follow from

$$C_1 C_6 \left(C_2^H C_7\right)^n < \left(C_8 C_9^n\right)^{-\delta},$$

which is the same as

$$\left(C_2^H C_7 C_9^\delta\right)^n < \left(C_1 C_6 C_8^\delta\right)^{-1}.$$

However, this inequality follows from (19) for n large enough.

Therefore, by the Subspace Theorem 2, there exist finitely many non-zero linear forms $\Lambda_1(\mathbf{X}), \dots, \Lambda_g(\mathbf{X})$ with coefficients in $\overline{\mathbb{Q}}$ such that each vector $\mathbf{x}_n$ is a zero of some Λ_j.

Now we have to consider two different cases depending on whether Λ_j depends on X_0 or not. Since the arguments are analogous, it is enough to consider $\Lambda := \Lambda_1$.

Suppose first Λ does not depend on X_0. Then, if $\Lambda(\mathbf{x}_n) = 0$, we have a nontrivial relation

$$u_0 + \sum_{i=1}^{h-1} u_i \beta_i^n = 0, \quad u_i \in \overline{\mathbb{Q}},\ i = 0, \dots, h-1.$$

By Lemma 1 this can hold for at most a finite number of n.

Suppose that Λ depends on X_0 and that $\Lambda(\mathbf{x}_n) = 0$. Then we have

$$z_n = v_0 + \sum_{i=1}^{h-1} v_i \beta_i^n, \quad v_i \in \overline{\mathbb{Q}},\ i = 0, \ldots, h-1. \tag{20}$$

Substituting this into $g_f(x_1, \ldots, x_r, z) = 0$ we get

$$g_f\left(\gamma_1^n, \ldots, \gamma_r^n, v_0 + \sum_{i=1}^{h-1} v_i \beta_i^n\right) = 0$$

and consequently

$$f\left(G_n^{(0)}, \ldots, G_n^{(d)}, v_0 \frac{\alpha^n}{{\alpha_1^{(0)}}^{\frac{n}{d}}} + \sum_{i=1}^{h-1} v_i \left(\frac{\beta_i \alpha}{{\alpha_1^{(0)}}^{\frac{1}{d}}}\right)^n\right) = 0.$$

This equation can either hold for infinitely many n (and in this case by the theorem of Skolem–Mahler–Lech for all n in a suitable arithmetic progression $\mathcal{P}$) or does not hold as an identity. The first case leads to the power sum

$$H_n = v_0 \frac{\alpha^n}{{\alpha_1^{(0)}}^{\frac{n}{d}}} + \sum_{i=1}^{h-1} v_i \left(\frac{\beta_i \alpha}{{\alpha_1^{(0)}}^{\frac{1}{d}}}\right)^n,$$

having the property that (n, H_n) is a solution for all n in the above arithmetic progression. Observe that H_n is in $\overline{\mathbb{Q}}\mathcal{E}_A$. But in the intersection of the arithmetic progressions $\{m(di) + k : m \in \mathbb{N}\} \cap \mathcal{P} =: \mathcal{P}_k$ for $0 \le k < di$ (where i is the index for which the maximum in the definition of α is obtained), the power sum H_n has positive rational roots. By taking conjugates, we immediately see that in fact the coefficients must be rationals too. But now by Lemma 1, page 322 in [4] it follows, since $H_n \in \mathbb{Z}$ for all n in the above intersection of arithmetic progressions, which is again an arithmetic progression, that

$$\tilde{H}_n := H_{n \in \mathcal{P}_k} \in \mathcal{E}_{\mathbb{Z}}^+.$$

The second case can only hold for finitely many n again by Lemma 1. Therefore the proof is finished. □

5. Proof of the corollaries

Let us notice that the assumption that the polynomial f is monic excludes the existence of families of solutions of the type (n, y) with arbitrary y (the second "trivial" infinite family in Theorem 1).

Proof of Corollary 1. In this case it is clear that we have

$$\alpha = \alpha_1^{(d-1)^{\frac{1}{d-1}}},$$

as dominating root in the equation

$$y^d + G_n^{(1)} y^{d-1} + \cdots + G_n^{(d)} = 0.$$

We therefore substitute

$$y = \alpha^n z.$$

Then all the roots of

$$\frac{1}{\alpha^{dn}} f(1, G_n^{(1)}, \ldots, G_n^{(d)}, \alpha^n z)$$

are ≤ 1, and the only roots $= 1$ appear as coefficients of y^d and y. Let $\gamma_1, \ldots, \gamma_r$ be all different roots < 1. Then we get

$$g_f(0, \ldots, 0, z) = z^d + a_1^{(d-1)} z = z(z^{d-1} + a_1^{(d-1)}),$$

which has only simple roots, and thus we can apply Theorem 1. From this the conclusion follows. □

Proof of Corollary 2. Our intention is to reduce the equation

$$y^3 + G_n^{(1)} y + G_n^{(2)} = 0$$

to equations for which Theorem 1 can be applied and from this we get the conclusion. For short we denote the dominating roots of $G_n^{(1)}$, $G_n^{(2)}$ by α, β, respectively. There are three cases to consider: $\alpha^3 < \beta^2$, $\alpha^3 > \beta^2$ or $\alpha^3 = \beta^2$.

Assume that $\alpha^3 > \beta^2$. We substitute $y = \alpha^{\frac{n}{2}} z$. Thus, we get

$$\alpha^{\frac{3n}{2}} z^3 + \left[a_1^{(1)} \alpha^{\frac{3n}{2}} + \cdots \right] z + a_1^{(2)} \beta^n + \cdots = 0,$$

where the roots in the dotted part are all $< \alpha^{\frac{3n}{2}}$. Dividing through by this number implies

$$z^3 + \left[a_1^{(1)} + \dots\right] z + a_1^{(2)} \left(\frac{\beta}{\alpha^{\frac{3}{2}}}\right)^n + \dots = 0.$$

Thus

$$g_f(0,0,z) = z\big(z^2 + a_1^{(1)}\big),$$

which has only simple roots and consequently Theorem 1 can be applied to this situation.

Now assume that $\beta^2 \geq \alpha^3$ holds. We substitute $y = \beta^{\frac{n}{3}} z$ and therefore get

$$\beta^n z^3 + \left[a_1^{(1)} \alpha^n \beta^{\frac{n}{3}} + \dots\right] z + a_1^{(2)} \beta^n + \dots = 0.$$

Dividing through by β^n gives

$$z^3 + \left[a_1^{(1)} \left(\frac{\alpha}{\beta^{\frac{2}{3}}}\right)^n + \dots\right] z + a_1^{(2)} + \dots = 0,$$

where again in the part not written down only roots < 1 appear. Now we have to consider the cases $\beta^2 > \alpha^3$ and $\beta^2 = \alpha^3$ separately.

If $\beta^2 > \alpha^3$ then we get

$$g_f(0,\dots,0,z) = z^3 + a_1^{(1)} z,$$

which has only simple roots and therefore Theorem 1 can be applied.

If $\beta^2 = \alpha^3$ we get

$$g_f(0,\dots,0,z) = z^3 + a_1^{(1)} z + a_1^{(2)},$$

and this cubic polynomial has only simple roots when

$$4{a_1^{(1)}}^3 + 27{a_1^{(2)}}^2 \neq 0$$

holds. Therefore the proof is finished. □

Poof of Corollary 3. First we have to consider, whether there exists $H_n \in \overline{\mathbb{Q}}\mathcal{E}_{\overline{\mathbb{Q}}}$ with

$$H_n^3 + a\alpha^n H_n + b\beta^n = 0$$

for all n in an arithmetic progression. Let H_n be a power sums with this property. Then it follows

$$H_n(H_n^2 + a\alpha^n) = -b\beta^n$$

and therefore H_n divides $b\beta^n$ in the ring $\overline{\mathbb{Q}}\mathcal{E}_{\overline{\mathbb{Q}}}$. Since the units of this ring are of the form $c\gamma^n$ we can conclude (since H_n divides a unit) that

$$H_n = c\gamma^n$$

with $c, \gamma \in \overline{\mathbb{Q}}$. In fact we have that γ is in the multiplicative group generated by α, β. Hence, we have to consider

$$c^3\gamma^{3n} + ac(\alpha\gamma)^n + b\beta^n = 0.$$

This equations has infinitely many solutions only in the case when $\gamma^3 = \beta$, $\gamma^2 = \alpha$ and $\alpha^3 = \beta^2$, because otherwise at least two elements of the set $\{\gamma^3, \alpha\gamma, \beta\}$ would be different and therefore the equation would have finitely many solutions by Lemma 1. In this case the equation becomes

$$(c^3 + ac + b)\beta^n = 0,$$

which either has no solution or is true identically if $c^3 + ac + b = 0$.

Thus, by Corollary 2, for $\alpha^3 \neq \beta^2$ the conclusion follows. In the case $\alpha^3 = \beta^2$ our equation reduces via the transformation

$$y = \beta^{\frac{n}{3}} z$$

and diving through by β^n to

$$z^3 + az + b = 0.$$

This in turn means that there are at most 3 possible values for z and each of them induces as above the (simple) power sums

$$\beta^{\frac{n}{3}} z_0.$$

Therefore, the proof is finished. □

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