Publ. Math. Debrecen
65/1-2 (2004), 173–177

On the divergence of partial sums of orthogonal series

By LÁSZLÓ LEINDLER (Szeged)

Dedicated to Professor László Csernyák on our 50 years friendship

Abstract. We slightly weaken the assumption of a theorem pertaining to the divergence of partial sums of orthogonal series from monotonicity to almost monotonicity.

1. Introduction

We take $(0,1)$ as the interval of orthogonality, and "almost everywhere" means simply in $(0,1)$ everywhere with the exception of at most a set of measure zero in the sense of Lebesgue.

Let $\{a_n\}$ be a given sequence of real numbers, and denote $\{m_n\}$ a fixed strictly increasing sequence of natural numbers. We put

$$A_n := \left\{a_{m_{n-1}}^2 + \cdots + a_{m_n}^2\right\}^{1/2} \quad (n = 1, 2, \ldots).$$

In a joint paper with L. Csernyák [1] we proved the following result, which is an improvement of a theorem due to K. Tandori [3].

Mathematics Subject Classification: Primary 42C15; Secondary 40G99.
Key words and phrases: orthogonal series, divergence of partial sums, Toeplitz summation process.
This research was partially supported by the Hungarian National Foundation for Scientific Research under Grant No. T042462.

Theorem A. *If $A_n \ge A_{n+1}$ and*

$$\sum_{n=2}^{\infty} A_n^2 \log^2 n = \infty, \tag{1.1}$$

then there exists a uniformly bounded orthonormal system $\{\psi_n(x)\}$ such that the m_n-th partial sums of the series

$$\sum_{k=1}^{\infty} a_k\, \psi_k(x) \tag{1.2}$$

are almost everywhere divergent.

The aim of this note is to extend Theorem A such that, instead of the monotonicity of $\{A_n\}$, only its almost monotonicity is required.

A nonnegative sequence $\mathbf{c} := \{c_n\}$ is called almost monotone nonincreasing if there exists a constant $K := K(\mathbf{c})$, depending on the sequence $\mathbf{c}$ only, such that for all $n \ge m$

$$c_n \le K\, c_m.$$

If a sequence $\mathbf{c}$ monotone nonincreasing, or almost monotone nonincreasing, we shall use the notations: $\mathbf{c} \in MS$ or $\mathbf{c} \in AMS$, respectively.

2. Results

Theorem. *Theorem A can be refined such that the condition $\{A_n\} \in MS$ is replaced by the assumption $\{A_n\} \in AMS$.*

With regard to a strictly increasing sequence $\mathbf{p} = \{p_n\}$ of natural numbers, we call a summability method $\mathbf{A}$ an $N(\mathbf{p})$ method if the following holds: In order that every orthogonal series $\sum_{n=1}^{\infty} c_n \varphi_n(x)$ with $\sum c_n^2 < \infty$ be summable $\mathbf{A}$ almost everywhere its p_n-th partial sums must converge almost everywhere. It is well known that every permanent Toeplitz summation process is an $N(\mathbf{p})$-summability with certain $\{p_n\}$. Our Theorem clearly implies the next result.

Corollary. *Let*

$$A_n^2(p) := \sum_{k=p_n+1}^{p_{n+1}} a_n^2.$$

If $\{A_n(p)\} \in AMS$ *and*

$$\sum_{n=2}^{\infty} A_n^2(p) \log^2 n = \infty,$$

then there exists a uniformly bounded orthonormal system $\{\psi_n(x)\}$ *such that the series* (1.2) *is not summable almost everywhere by some* $N(\mathbf{p})$ *method.*

We mention that, by virtue of a former theorem of the author ([2], Satz II) Theorem and Corollary can be improved both such that the system $\{\psi_n(x)\}$ is replaced by a uniformly bounded polynom system $\{P_n(x)\}$.

3. Lemmas

We shall use the following two lemmas. The first lemma is proved implicitly in [1].

Lemma 1. *Under the assumptions of Theorem A there exist an index-sequence* $N_0 < N_1 \dots < N_m < \dots$, *a uniformly bounded orthonormal system* $\{\psi_n(x)\}$ *and a sequence of simple sets* H_k *(*H_k *is the union of finite intervals) such that*

(i) *for every* $x \in H_k$ *there is some* $n_k(x) \in \mathbb{N}$ *such that*

$$\left| \sum_{i=N_k}^{N_k+n_k(x)} \sum_{\ell=m_{i-1}+1}^{m_i} a_\ell \, \psi_\ell(x) \right| \ge D, \quad (k = 1, 2, \dots), \tag{3.1}$$

where D *is a positive constant, and the sums*

$$s_i(x) := \sum_{\ell=m_{i-1}+1}^{m_i} a_\ell \, \psi_\ell(x) \quad (i = N_k, \dots, N_k + n_k(x)) \tag{3.2}$$

have equal signs,

(ii) *the sets* H_k $(k = 0, 1, \dots)$ *are stochastically independent and*

$$\sum_{k=0}^{\infty} \mu(H_k) = \infty, \tag{3.3}$$

where $\mu(H)$ *denotes the Lebesgue measure of* H.

Lemma 2. *If* $\mathbf{c} := \{c_n\} \in AMS$ *and* $\gamma_n := \sup_{k \geq n} c_k$, *then*

$$c_n \leq \gamma_n \leq K(\mathbf{c})c_n \tag{3.4}$$

holds.

Proof. By $\mathbf{c} \in AMS$

$$\gamma_n \leq \sup_{k \geq n} K(\mathbf{c})c_n = K(\mathbf{c})c_n,$$

this and the definition of γ_n clearly yield (3.4).

4. Proof of Theorem

Denote $\alpha := \{A_n\}$ and let $A_n^* := \sup_{k \geq n} A_k$. Then clearly $\{A_n^*\} \in MS$, and by Lemma 2

$$A_n \leq A_n^* \leq K(\alpha)A_n \tag{4.1}$$

holds. Thus, for $\rho_n := \frac{A_n^*}{A_n}$ we have

$$1 \leq \rho_n \leq K(\alpha). \tag{4.2}$$

Moreover by (1.1) and (4.1)

$$\sum_{n=2}^{\infty} (A_n^*)^2 \log^2 n = \infty. \tag{4.3}$$

Next let us define a new sequence $\{a_k^*\}$ as follows:

$$a_k^* := \rho_n a_k \quad \text{if } m_{n-1} < k \leq m_n, \quad n = 1, 2, \dots$$

Then

$$\sum_{k=m_{n-1}+1}^{m_n} (a_k^*)^2 = \rho_n^2 \sum_{k=m_{n-1}+1}^{m_n} a_k^2 = \rho_n^2 A_n^2 = (A_n^*)^2.$$

Therefore we can apply Lemma 1 with the sequence $\{a_k^*\}$ and obtain that (3.1) holds with $\{a_k^*\}$ in place of $\{a_k\}$, furthermore the sums in (3.2) with $\{a_\ell^*\}$ have equal signs for all i. Using these facts we get that

$$\left|\sum_{i=N_k}^{N_k+n_k(x)} \sum_{\ell=m_{i-1}+1}^{m_i} a_\ell^*\, \psi_\ell(x)\right| = \left|\sum_{i=N_k}^{N_k+n_k(x)} \rho_i \sum_{\ell=m_{i-1}+1}^{m_i} a_\ell\, \psi_\ell(x)\right| \geq D,$$

whence, by (4.2),

$$\left|\sum_{i=N_k}^{N_k+n_k(x)} \sum_{\ell=m_{i-1}+1}^{m_i} a_\ell\, \psi_\ell(x)\right| \geq \frac{D}{K(\alpha)} > 0 \tag{4.4}$$

follows.

In virtue of (3.3) and the Borel–Cantelli lemma we get that

$$\mu\Big(\overline{\lim_{k\to\infty}} H_k\Big) = 1,$$

that is, almost every $x \in (0,1)$ belongs to $\overline{\lim}_{k\to\infty} H_k$. Thus (4.3) holds almost everywhere for infinite k.

Consequently the m_n-th partial sums of the series (1.2) are almost everywhere divergent.

This completes the proof. □

References

[1] L. Csernyák und L. Leindler, Über die Divergenz der Partialsummen von Orthogonalreihen, *Acta Sci. Math. (Szeged)* **27** (1966), 55–61.

[2] L. Leindler, Über die orthogonalen Polynomsysteme, *Acta Sci. Math. (Szeged)* **21** (1960), 19–46.

[3] K. Tandori, Über die Divergenz der Orthogonalreihen, *Publ. Math. (Debrecen)* **8** (1961), 291–307.

LÁSZLÓ LEINDLER
BOLYAI INSTITUTE
UNIVERSITY OF SZEGED
ARADI VÉRTANÚK TERE 1
H-6720 SZEGED
HUNGARY

(Received April 23, 2003; revised January 27, 2004)