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Mahler's classification of numbers compared with Koksma's, III

By YANN BUGEAUD (Strasbourg)

To the memory of Professor B. Brindza

Abstract. Let $n \geq 1$ be an integer. In the 1930's, MAHLER and KOKSMA defined on the set $\mathbb{C}$ of complex numbers the functions w_n and w_n^*, respectively, and used them to classify $\mathbb{C}$ into four classes. It turns out that both classifications are equivalent. However, when $n \geq 2$, there exist complex numbers ξ for which $w_n(\xi)$ and $w_n^*(\xi)$ are different. In the present note, we prove that the inequalities $0 \leq w_2(\xi) - w_2^*(\xi) \leq 1$ and $0 \leq w_3(\xi) - w_3^*(\xi) \leq 2$ are essentially best possible.

1. Introduction

Let $n \geq 1$ be an integer and ξ be a complex number. We denote by $w_n(\xi)$ the supremum of the exponents w for which

$$0 < |P(\xi)| < \mathrm{H}(P)^{-w}$$

has infinitely many solutions in integer polynomials $P(X)$ of degree at most n. Furthermore, we denote by $w_n^*(\xi)$ the supremum of the exponents w^* for which

$$0 < |\xi - \alpha| < \mathrm{H}(\alpha)^{-w^*-1}$$

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has infinitely many solutions in complex algebraic numbers α of degree at most n. Throughout the present note, $\mathrm{H}(P)$ stands for the naïve height of the polynomial $P(X)$ (that is, the maximum of the absolute values of its coefficients) and $\mathrm{H}(\alpha)$ stands for the naïve height of α (that is, the naïve height of its minimal defining polynomial over $\mathbb{Z}$). The functions w_n and w_n^* have been introduced in 1932 by Mahler [7] and in 1939 by Koksma [6], respectively, in order to classify the set of complex numbers.

Clearly, the functions w_1 and w_1^* coincide, and it is quite easy to show (see e.g. [4, Section 3.4]) that we have

$$0 \le w_n(\xi) - w_n^*(\xi) \le n - 1. \tag{1}$$

Furthermore, Sprindžuk [11] established that $w_n(\xi) = w_n^*(\xi) = n$ holds for all $n \ge 1$ and almost all ξ (in the sense of the Lebesgue measure on the complex plane). In 1976, R. C. Baker [1] proved that there exist real numbers ξ such that $w_n^*(\xi) < w_n(\xi)$ for some integer $n \ge 2$. More precisely, he established that for any integer $n \ge 2$ the function $w_n - w_n^*$ can take any value in the interval $[0, (n-1)/n]$. This has been subsequently improved upon by Bugeaud [2], who showed that, for any integer $n \ge 3$, the set of values taken by the function $w_n - w_n^*$ contains the interval $[0, n/4]$ (see also [3] for an improvement when $n \ge 6$ is even). Like Baker's, the approach followed in [2] originates in two papers by Schmidt [9], [10], where the existence of T-numbers is established (these are transcendental numbers ξ for which $\limsup_{n\to+\infty} w_n(\xi)/n = +\infty$ and $w_n(\xi)$ is finite for any $n \ge 1$). The main novelty introduced in [2] is the use in the inductive construction of integer polynomials having two zeros very close to each other.

In the present note, we restrict our attention to the cases $n = 2$ and $n = 3$. We show that, in both cases, inequalities (1) are essentially best possible. Our proof rests on the construction of families of quadratic and cubic irreducible, integer polynomials having two real roots very close to each other. It makes use of a recent result of Evertse [5].

2. Results

Baker [1] and Bugeaud [2] established that the functions $w_2 - w_2^*$ and $w_3 - w_3^*$ take any value in the intervals $[0, 1/2]$ and $[0, 3/4]$, respectively. Theorem 1 below extends both results.

Theorem 1. *The set of values taken by the function $w_2 - w_2^*$ contains the interval $[0, 1)$. The set of values taken by the function $w_3 - w_3^*$ contains the interval $[0, 2)$.*

In view of (1), Theorem 1 is essentially best possible. Unfortunately, we are unable to decide whether there exists a real number ξ satisfying $w_2(\xi) = w_2^*(\xi) + 1$ or $w_3(\xi) = w_3^*(\xi) + 2$.

We point out that, in the quadratic case, the proof of Theorem 1 is effective. However, in the cubic case, it is ineffective, since it ultimately depends on Roth's Theorem.

The proof of Theorem 1 follows the same general strategy as that of the main results from [2] and [3]. The key point is the existence of families of quadratic and cubic irreducible, integer polynomials having two real roots very close to each other, given by Lemmas 1 and 3 below. Since these families of polynomials are not parametrized in the same way as the families of polynomials used in [2], we have to proceed slightly differently than in [2]. Furthermore, additional complication occurs since our family of cubic polynomials is not explicitly given.

In order to shorten the length of the present note, we choose not to give a full proof of Theorem 1. We merely refer the reader to [2] and explain with full details which modifications are required to adapt the proof of the main result from [2] in order to get Theorem 1.

Our method also allows us to construct real numbers ξ with prescribed values for $w_1(\xi)$, $w_2(\xi)$, $w_3(\xi)$, $w_2^*(\xi)$ and $w_3^*(\xi)$. Suitable modification of the proof of Theorem 1 yields the following result.

Theorem 2. *Let w_1, w_2, w_3, w_2^* and w_3^* be sufficiently large real numbers satisfying*

$$w_1 \le w_2 \le w_3, \quad w_2^* \le w_3^*, \quad 0 \le w_2 - w_2^* < 1, \quad \text{and} \quad 0 \le w_3 - w_3^* < 2.$$

Then, there exists a real number ξ with

$$w_i(\xi) = w_i \ (i = 1, 2, 3) \quad \text{and} \quad w_i^*(\xi) = w_i^* \ (i = 2, 3).$$

Theorem 2 provides a new contribution to the resolution of Problem 1 from [4]. We omit its proof, which does not require any new idea (see Theorem 2 from [2]).

3. Auxiliary results

Throughout this Section, we write $A \ll B$ if there exists an absolute positive constant c such that $|A| < cB$, and we write $A \asymp B$ if both $A \ll B$ and $B \ll A$ hold.

Let $n \geq 2$ be an integer and $P(X)$ be an integer polynomial of degree n without multiple roots. Denote by $\gamma_1, \ldots, \gamma_n$ the roots of $P(X)$. It is well-known (see e.g. [5]) that there exists a positive constant $c(n)$, depending only on n, such that for any subset Σ of $\{1, \ldots, n\}$ of cardinality $|\Sigma| \geq 2$ we have

$$\prod_{\{i,j\} \subset \Sigma} |\gamma_i - \gamma_j| \geq c(n)\, \mathrm{H}(P)^{1-n}. \tag{2}$$

This estimate is best possible when $|\Sigma| = n$ (see e.g. Theorem 1.2 from [5]) and for $|\Sigma| = 2$ and $n = 3$ (see [8]). Namely, let $(p_m/q_m)_{m \geq 1}$ denote the sequence of convergents to $\sqrt{2}$. The integer polynomial $P_m(X) := (X^2 - 2)(q_m X - p_m)$ has two roots $\gamma_1 = \sqrt{2}$ and $\gamma_2 = p_m/q_m$ satisfying

$$|\gamma_1 - \gamma_2| \asymp \frac{1}{\mathrm{H}(P_m)^2}.$$

Notice, however, that $P_m(X)$ is reducible over $\mathbb{Z}$. The same idea can be used to prove that (2) is best possible for any $n \geq 4$ and $|\Sigma| = n - 1$ or n. It suffices to consider the polynomials

$$(X^2 - 2)(q_m X - p_m) \ldots (q_{m+n-3} X - p_{m+n-3})$$

and

$$(q_m X - p_m) \ldots (q_{m+n-1} X - p_{m+n-1}).$$

We point out that the roots of the polynomials from the above families are uniformly bounded (unlike in the construction given in [5]).

When $|\Sigma| = 2$, Lemma 1 and Lemma 2 below show that (2) is essentially best possible for $n = 2$ and $n = 3$, respectively, under the extra

assumption that $P(X)$ is irreducible and has only real roots. Presumably, Lemma 1 is not new. However, we were unable to find it in the literature.

Lemma 1. *Let (a,b) with a and b positive integers be a solution to the Pellian equation $X^2-8Y^2=17$. Then the roots γ_1 and γ_2 of the polynomial $P_{a,b}(X):=bX^2-aX+2b$ satisfy*

$$|\gamma_1-\gamma_2| \asymp \frac{1}{\mathrm{H}(P_{a,b})}.$$

Proof. This is an easy verification. □

Lemma 1 allows us to construct explicitly a family of quartic polynomials without rational roots having three roots very close to each other. Indeed, for a and b as in Lemma 1, set

$$R_{a,b}(X)=(X^2-2)(bX^2-aX+2b).$$

The roots of $R_{a,b}(X)$ are

$$\gamma_1=\sqrt{2},\quad \gamma_2=-\sqrt{2},\quad \gamma_3=\sqrt{2}\sqrt{1+\frac{17}{8b^2}}+\frac{\sqrt{17}}{2b},\qquad \text{and}$$
$$\gamma_4=\sqrt{2}\sqrt{1+\frac{17}{8b^2}}-\frac{\sqrt{17}}{2b}.$$

It is easy to check that

$$|\gamma_1-\gamma_3|\cdot|\gamma_1-\gamma_4|\cdot|\gamma_3-\gamma_4| \asymp \frac{1}{\mathrm{H}(R_{a,b})^3},$$

showing that (2) is best possible for $n=4$ and $|\Sigma|=3$. This construction can be extended to any degree. Indeed, let $(a_1,b_1),\dots,(a_\ell,b_\ell)$ be consecutive solutions to the Pellian equation $X^2-8Y^2=17$. Let p/q denote a convergent to $\sqrt{2}$ with $p \asymp \sqrt{a_1}$. By considering the polynomials

$$(X^2-2)(b_1X^2-a_1X+2b_1)\dots(b_\ell X^2-a_\ell X+2b_\ell)$$

and

$$(qX-p)(X^2-2)(b_1X^2-a_1X+2b_1)\dots(b_\ell X^2-a_\ell X+2b_\ell),$$

one sees again that (2) is best possible for any integer $n\ge 5$ and $|\Sigma|=n-1$.

Lemma 2. *Let ε be a positive real number. Then, there exists a cubic integer, primitive, irreducible polynomial $P(X) = a(X-\gamma_1)(X-\gamma_2)\times(X-\gamma_3)$ with $a > 0$, of arbitrarily large height, satisfying*

$$\frac{1}{\mathrm{H}(P)^2} \ll |\gamma_1 - \gamma_2| \ll \frac{1}{\mathrm{H}(P)^{2-\varepsilon}},$$

$$a \gg \mathrm{H}(P)^{1-\varepsilon} \quad \text{and} \quad |\gamma_1|, |\gamma_2|, |\gamma_3| \ll \mathrm{H}(P)^{\varepsilon}.$$

Proof. This is established with slightly different notation in the 'Proof of part (i) of Theorems 1.1 and 1.2' of Evertse [5]. An easy computation gives the upper bound for the $|\gamma_i|$'s and the lower bound for a. □

We point out that the proof of Lemma 2 ultimately depends on Roth's Theorem, hence, it is of an ineffective nature.

Incidentally, Lemmas 1 and 2 imply that Lemma A.8 from [4] is best possible for $n = 2$ and $n = 3$, as far as the dependence on the height of the polynomial is concerned.

For the proof of Theorem 1, we need the following consequence of Lemma 2.

Lemma 3. *There exist a sequence $(P_m(X))_{m\ge1}$ of primitive, irreducible, integer, cubic polynomials $P_m(X) = a_m(X-\gamma_m)(X-\gamma_m^\sigma)(X-\gamma_m^\tau)$ with real roots, a strictly decreasing sequence $(\varepsilon_m)_{m\ge1}$ of positive real numbers, two sequences $(\eta_m)_{m\ge1}$ and $(\eta'_m)_{m\ge1}$ of positive real numbers such that $a_m > 0$, $\mathrm{H}(P_{m+1}) > \mathrm{H}(P_m)$, $\eta_m \le \varepsilon_m$, $\eta'_m \le \varepsilon_m$,*

$$a_m\,|\gamma_m - \gamma_m^\sigma| \cdot |\gamma_m - \gamma_m^\tau| = \mathrm{H}(P_m)^{-1+\eta_m}, \quad a_m = \mathrm{H}(P_m)^{1-\eta'_m}$$

and

$$|\gamma_m|, |\gamma_m^\sigma|, |\gamma_m^\tau| \le \mathrm{H}(P_m)^{\varepsilon_m},$$

for any $m \ge 1$.

Proof. This follows from Lemma 2 combined with the well-known upper bound $a_m|\gamma_m - \gamma_m^\tau| \ll \mathrm{H}(P_m)$. □

We conclude this Section with an easy (but useful) lemma.

Lemma 4. *Let n be a positive integer and let g be a prime number with $g > n$. Let $P(X)$ be a primitive, integer polynomial of degree n. If g does not divide the leading coefficient of $P(X)$, then there is no integer c such that g divides each of $P(c), P(c+1), \ldots, P(c+n)$.*

PROOF. This is a straightforward extension of Lemma 4 of BAKER [1]. □

4. Proof of Theorem 1

Let n be 2 or 3. Let μ be a positive real number. Throughout this Section, we write $A \ll B$ if there exists a positive constant $c(\mu)$, depending only on μ, such that $|A| < c(\mu)B$, and we write $A \asymp B$ if both $A \ll B$ and $B \ll A$ hold.

Our aim is to construct inductively a converging sequence $(\xi_j)_{j\geq 1}$ of real algebraic numbers of degree n. As in [2], the ξ_j's are of the shape

$$\xi_j = \frac{c_j + \gamma_j}{g_j},$$

where g_j is a prime number, c_j a positive integer, and γ_j a suitable real algebraic number of degree n. More precisely, we have to establish the following proposition.

Proposition 1. *Let χ be a sufficiently large real number. Let $(\nu_j)_{j\geq 1}$ be a sequence of real numbers ≥ 2. Then, there exist a positive real number $\lambda < 1/2$, totally real algebraic numbers $\gamma_1, \gamma_2, \ldots$ of degree n, prime numbers $g_1 \geq 11$, $g_2, \ldots$ and integers $c_1, c_2, \ldots$ such that the following conditions are satisfied:*

(I_j) *g_j does not divide the norm of $c_j + \gamma_j$ $(j \geq 1)$.*

(II_1) $$\xi_1 = (c_1 + \gamma_1)/g_1 \in]1, 2[\,.$$

(II_j) *$\xi_j = (c_j + \gamma_j)/g_j$ belongs to the interval I_{j-1} defined by*

$$\xi_{j-1} + \frac{1}{2}g_{j-1}^{-\nu_{j-1}} < x < \xi_{j-1} + \frac{3}{4}g_{j-1}^{-\nu_{j-1}} \quad (j \geq 2).$$

(III_1) $$|\xi_1 - \alpha| \geq 2\lambda\, \mathrm{H}(\alpha)^{-\chi}$$

for any algebraic number $\alpha \neq \xi_1$ of degree $\leq n$.

(III_j) $$|\xi_j - \alpha| \geq \lambda\, \mathrm{H}(\alpha)^{-\chi}$$

for any algebraic number $\alpha \notin \{\xi_1, \ldots, \xi_j\}$ of degree $\leq n$ $(j \geq 2)$.

In addition to (I_j) to (III_j), we also require that γ_j, g_j and μ are strongly connected. In [2] and [3], the auxiliary algebraic numbers γ_j are given in terms of the parameter μ. Here, we proceed alternatively: the γ_j's will be roots of polynomials given by Lemmas 1 and 3, and g_j will be defined in terms of γ_j and μ. This is slightly more technical than in the previous papers. The reason for that is that we have at our disposal sets of polynomials parametrized in a different way. Furthermore, in [2] and [3], the sequence $(\nu_j)_{j\ge 1}$ is assumed to be constant. However, what is really required in the proof is that the sequence $(g_j^{\nu_j})_{j\ge 1}$ is increasing. In the sequel, $(\nu_j)_{j\ge 1}$ will be constant in the case $n = 2$, but it is crucial to allow it to vary when dealing with the case $n = 3$.

Let $P_j(X)$ denote the minimal polynomial of γ_j. The only aim of Condition (I_j) is to ensure that the polynomial $Q_j(X) := P_j(g_jX - c_j)$ is primitive. At this point, we use Lemma 4. This is the reason why we impose to the g_j's to be prime numbers.

Consider first the case $n = 2$. Let Δ be real with $1/2 < \Delta < 1$, and set $\mu = (2\Delta - 1)/(1 - \Delta)$. Observe that $0 < \mu < +\infty$. Let $(\tilde{a}_m, \tilde{b}_m)$, $m \ge 1$, be the sequence of all positive solutions to the Pellian equation $X^2 - 8Y^2 = 17$ numbered such that $(\tilde{a}_1, \tilde{b}_1) = (5, 1)$ and $\tilde{a}_{m+1} > \tilde{a}_m$ for any positive integer m.

For any sufficiently large m, let $\tilde{g}_m$ be a prime number with

$$\tilde{g}_m \asymp \tilde{a}_m^{1/\mu} \asymp \tilde{b}_m^{1/\mu} \quad \text{and} \quad \gcd(\tilde{g}_m, \tilde{b}_m) = 1, \tag{3}$$

and denote by $\tilde{\gamma}_m$ the largest root of the polynomial $\tilde{P}_m(X) := \tilde{b}_m X^2 - \tilde{a}_m X + 2\tilde{b}_m$. The existence of $\tilde{g}_m$ is a consequence of Bertrand's Postulate and the fact that $\tilde{b}_m$ is divisible by at most $[\mu] + 1$ distinct prime numbers of size $\asymp \tilde{b}_m^{1/\mu}$. Observe that we have $0 < \tilde{\gamma}_m < 5$. Now, we show how to extract from $((\tilde{g}_m, \tilde{\gamma}_m))_{m\ge 1}$ a suitable subsequence $((g_j, \gamma_j))_{j\ge 1}$.

Going through the proof of Theorem 3 from [2], we see that, in order to satisfy conditions (II_j) and (III_j), we 'only' need to take, at each step, g_{j+1} sufficiently large compared with g_j. Since $(\tilde{g}_m)_{m\ge 1}$ tends to infinity, this can easily be done. One should, however, observe that inequalities (8) from [2] do not hold anymore, since $\mathrm{H}(\gamma_j)$ can be large compared with g_j. The only consequence of that is that χ must be large (a lower bound for it can be expressed in terms of μ, however, for sake of simplicity, we do not

give details). In view of conditions (II_j) and since the γ_j's are uniformly bounded, the integers c_j and g_j are of comparable size.

It remains for us to deal with conditions (I_j). The minimal defining polynomial of ξ_j is

$$Q_j(X) := P_j(g_jX - c_j) = b_jg_j^2X^2 - (4b_jc_j + a_j)g_jX + 2b_jc_j^2 + 2b_j + a_jc_j,$$

and Lemma 4 ensures that there are many suitable choices for c_j (we should, however, impose that g_j does not divide c_j) such that $Q_j(X)$ is primitive. Combined with (3), this shows that

$$\mathrm{H}(Q_j) \asymp \mathrm{H}(\xi_j) \asymp g_j^{2+\mu}. \tag{4}$$

Let w_2^* and χ be sufficiently large real numbers with $\chi < w_2^* + 1$. Set $\nu_j = (w_2^* + 1)(2 + \mu)$ for any $j \geq 1$. To summarize, we have explained how to construct inductively a sequence of quadratic numbers $(\xi_j)_{j\geq 1}$ satisfying the conclusion of Proposition 1. It converges to a limit that we denote by ξ. Since for any $j \geq 1$ we have

$$g_j^{-\nu_j}/2 \leq |\xi - \xi_j| \leq g_j^{-\nu_j}, \tag{5}$$

we deduce from (4) that

$$|\xi - \xi_j| \asymp \mathrm{H}(\xi_j)^{-\nu_j/(2+\mu)} \asymp \mathrm{H}(\xi_j)^{-w_2^*-1}. \tag{6}$$

Combined with (III_j) and our choice of χ, this proves that $w_2^*(\xi) = w_2^*$.

Denoting by ξ_j^σ the conjugate of ξ_j and by γ_j^σ the one of γ_j, we get from (3)–(6) and Lemma 1 that

$$\begin{aligned}
|Q_j(\xi)| &= b_jg_j^2 \cdot |\xi - \xi_j| \cdot |\xi - \xi_j^\sigma| \\
&\asymp g_j^{2+\mu} \cdot \left(\xi - \frac{c_j + \gamma_j}{g_j}\right) \left(\xi - \frac{c_j + \gamma_j^\sigma}{g_j}\right) \\
&\asymp g_j^{2+\mu} \cdot \left(\xi - \frac{c_j + \gamma_j}{g_j}\right) \left(\xi_j - \frac{c_j + \gamma_j^\sigma}{g_j}\right) \\
&\asymp g_j^{2+\mu} \cdot \left(\xi - \frac{c_j + \gamma_j}{g_j}\right) \left(\frac{\gamma_j - \gamma_j^\sigma}{g_j}\right) \\
&\asymp g_j \cdot \mathrm{H}(\xi_j)^{-w_2^*-1} \asymp \mathrm{H}(\xi_j)^{-w_2^*-1+1/(2+\mu)}.
\end{aligned}$$

Consequently, we obtain

$$w_2(\xi) \geq w_2^*(\xi) + 1 - \frac{1}{2+\mu} = w_2^*(\xi) + \Delta, \tag{7}$$

by the definition of μ. The fact that we have indeed an equality in (7) can be proved exactly as in [1] or [2].

Consider now the case $n = 3$. Let Δ be a real number with $2/3 < \Delta < 2$, and set $\mu = (3\Delta - 2)/(2 - \Delta)$. Observe that $0 < \mu < +\infty$. Let w_3^* and χ be sufficiently large real numbers with $\chi < w_3^* + 1$. Let $P(X)$ be a primitive, irreducible, integer polynomial belonging to the sequence given by Lemma 3 (we drop the index j). Denote by $a > 0$ its leading coefficient and by γ any one of its roots. Let ε, η, and η' be the positive real numbers associated to $P(X)$ as in Lemma 3.

Our first aim is to construct a suitable prime number g and a suitable integer c such that the polynomial $Q(X) = P(gX - c)$ is primitive. Observe that the leading coefficient of $Q(X)$ is $ag^3 = \mathrm{H}(P)^{1-\eta'} g^3$. If ε is sufficiently small, then there exists a real number g_0 satisfying

$$g_0 \mathrm{H}(P)^{-1+\eta} = \left(\mathrm{H}(P)^{1-\eta'} g_0^3\right)^{(1-\mu)/(3+\mu)}.$$

Furthermore, by selecting ε small enough (this means, by taking, if needed, another polynomial $P(X)$ in the family given by Lemma 3), g_0 can be made arbitrarily large and we can, in addition, ensure that $g_0 \gg \mathrm{H}(P)^{2\varepsilon}$. This implies that $|\gamma| \leq g_0^{1/2}$ and $\mathrm{H}(P([g_0]X - c)) = a[g_0]^3$, since $a \gg \mathrm{H}(P)^{1-\varepsilon}$. Using Bertrand's Postulate as in the case $n = 2$, we find a prime number g which does not divide a and satisfies $g \asymp g_0$. Consequently, we get

$$g\mathrm{H}(P)^{-1+\eta} \asymp \left(\mathrm{H}(P)^{1-\eta'} g^3\right)^{(1-\mu)/(3+\mu)} \asymp \mathrm{H}(Q)^{(1-\mu)/(3+\mu)}$$

and $\mathrm{H}(Q) \asymp \mathrm{H}(P)^{1-\eta'} g^3$.

Before going on, we point out that if it would have been possible to take $\eta = \eta' = 0$ (this happens in the case $n = 2$), then we would have get $g \asymp \mathrm{H}(P)^{1/\mu}$ and $\mathrm{H}(Q) \asymp g^{3+\mu}$.

Consequently, using a subsequence of the sequence of polynomials given by Lemma 3, we construct a sequence of quadruples $((g_j, c_j, \gamma_j, \nu_j))_{j \geq 1}$ with g_{j+1} sufficiently large compared with g_j for any

$j \geq 1$, and such that

$$\begin{aligned} g_j \mathrm{H}(P_j)^{-1+\eta_j} &\asymp \mathrm{H}(Q_j)^{(1-\mu)/(3+\mu)}, \\ |\gamma_j| \leq g_j^{1/2} \quad &\text{and} \quad \mathrm{H}(Q_j)^{w_3^*+1} = g^{\nu_j}, \end{aligned} \tag{8}$$

where $P_j(X)$ is the minimal defining polynomial of γ_j and $Q_j(X) = P_j(g_jX - c_j)$. Observe that g_j and c_j are of comparable size, since $|\gamma_j| \leq g_j^{1/2}$. As in the case $n = 2$, we use Lemma 4 to deal with Condition (I_j). We skip the details.

To summarize, we have constructed inductively a sequence of cubic numbers $(\xi_j)_{j\geq 1}$ satisfying the conclusion of Proposition 1. It converges to a limit that we denote by ξ. Since for any $j \geq 1$ we have

$$g_j^{-\nu_j}/2 \leq |\xi - \xi_j| \leq g_j^{-\nu_j}, \tag{9}$$

we deduce from (8) that

$$|\xi - \xi_j| \asymp \mathrm{H}(\xi_j)^{-w_3^*-1}. \tag{10}$$

Denoting by γ_j^σ, γ_j^τ, ξ_j^σ, and ξ_j^τ the conjugates of γ_j and ξ_j, we infer from (8)–(10) and Lemma 3 that

$$\begin{aligned} |Q_j(\xi)| &= a_j g_j^3 \cdot |\xi - \xi_j| \cdot |\xi - \xi_j^\sigma| \cdot |\xi - \xi_j^\tau| \\ &\asymp a_j g_j^3 \cdot \left(\xi - \frac{c_j + \gamma_j}{g_j}\right) \left(\xi_j - \frac{c_j + \gamma_j^\sigma}{g_j}\right) \left(\xi_j - \frac{c_j + \gamma_j^\tau}{g_j}\right) \\ &\asymp a_j g_j^3 \cdot \left(\xi - \frac{c_j + \gamma_j}{g_j}\right) \left(\frac{\gamma_j - \gamma_j^\sigma}{g_j}\right) \left(\frac{\gamma_j - \gamma_j^\tau}{g_j}\right) \\ &\asymp g_j \cdot \mathrm{H}(\xi_j)^{-w_3^*-1} \cdot \mathrm{H}(P_j)^{-1+\eta_j} \asymp \mathrm{H}(\xi_j)^{-w_3^*-1+(1-\mu)/(3+\mu)}. \end{aligned}$$

Consequently, we obtain

$$w_3(\xi) \geq w_3^*(\xi) + 1 - \frac{1-\mu}{3+\mu} = w_3^*(\xi) + \Delta, \tag{11}$$

by the definition of μ. The fact that we have indeed an equality in (11) can be proved exactly as in [1] or [2].

Thus, we have established that the set of values taken by $w_2 - w_2^*$ (resp. by $w_3 - w_3^*$) includes the interval $(1/2, 1)$ (resp. the interval $(2/3, 2)$). Combined with the results from [1] and [2] recalled just before the statement of Theorem 1, this completes the proof of our theorem. □

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YANN BUGEAUD
UNIVERSITÉ LOUIS PASTEUR
U. F. R. DE MATHÉMATIQUES
7, RUE RENÉ DESCARTES
67084 STRASBOURG
FRANCE

E-mail: bugeaud@math.u-strasbg.fr

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