Publ. Math. Debrecen
65/3-4 (2004), 341–362

Power values of polynomials and binomial Thue–Mahler equations

By K. GYŐRY (Debrecen), I. PINK (Debrecen) and
A. PINTÉR (Debrecen)

To the memory of Professors B. Brindza and J. Erdős

Abstract. Let $p_1, \dots, p_s$ be distinct primes, and S the set of integers not divisible by primes different from $p_1, \dots, p_s$. We give effectively computable upper bounds for n in the equations (1) $f(x) = wy^n$, (5) $F(x, z) = wy^n$ and (9) $ax^n - by^n = c$, where $f \in \mathbf{Z}[X]$ is a monic polynomial, $F \in \mathbf{Z}[X, Z]$ a monic binary form, the discriminants $D(f)$, $D(F)$ are contained in S, and x, y, z, w, a, b, c, n are unknown non-zero integers with $z, w, a, b, c \in S$, $y \notin S$ and $n \geq 3$. It is a novelty in our paper that the upper bounds depend only on the product $p_1 \cdots p_s$ and, in case of (1) and (5), on deg f and deg F, respectively. The bounds are given explicitly in terms of $p_1 \cdots p_s$. Our results are established in more general forms, over an arbitrary algebraic number field. Equation (5) is reduced to an equation of type (9) over an appropriate extension of the ground field. The proofs involve among other things the best known estimates for linear forms in logarithms of algebraic numbers.

Mathematics Subject Classification: 11D61, 11D57.
Key words and phrases: superelliptic equations, binomial Thue–Mahler equations, S-unit equations, discriminants of polynomials and binary forms.
The first and third authors were supported by the Hungarian Academy of Sciences and by grants 38225 and 42985 from the Hungarian National Foundation for Scientific Research. The second and third authors were supported by grant F034981 of the Hungarian National Foundation for Scientific Research and by the FKFP grant 3272-13/066/2001.

1. Introduction

Superelliptic equations. Let f be a polynomial with integer coefficients and with at least two distinct zeros, and let w be a given non-zero integer. Schinzel and Tijdeman [19] proved that if the integers x, y, n with $|y| > 1$ and $n \geq 2$ satisfy the equation

$$f(x) = wy^n, \tag{1}$$

then n can be bounded above by an effectively computable number depending only on f and w. Several upper bounds were later obtained for n which depend on w and the height and degree of f; see [21], [4], [22], [2], [5], [1], [17] and the references given there. Some of these results were established in more general form, over number fields and/or assuming only on w that its distinct prime factors are fixed.

For $w = 1$ and irreducible monic f, Brindza, Evertse and Győry [3] derived an explicit upper bound for n which depends only on the degree and discriminant, $D(f)$, of f. Recently Haristoy [14] extended this to arbitrary monic polynomials as well as to the number field case.

In our paper we show that, apart from certain exceptions which will be described explicitly below, n can be estimated from above in (1) by an effectively computable bound which depends only on deg f and the product of distinct prime factors of w and $D(f)$. We prove this in a more general form, over number fields.

To formulate our results, we have to introduce some notation. Throughout this paper, $\mathbf{K}$ denotes an algebraic number field of degree d with ring of integers $O_{\mathbf{K}}$ and unit group $O_{\mathbf{K}}^*$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ ($s \geq 0$) be distinct prime ideals of $O_{\mathbf{K}}$ and denote by S the set of those $\alpha \in O_{\mathbf{K}} \setminus \{0\}$ for which the ideal (α) has no prime ideal divisors other than $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$. Further, let

$$Q = \begin{cases} N_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p}_1 \cdots \mathfrak{p}_s) & \text{if } s > 0, \\ 1 & \text{if } s = 0. \end{cases}$$

Let $f \in O_{\mathbf{K}}[X]$ denote in (1) a monic polynomial of degree m with $k \geq 2$ distinct zeros and splitting field $\mathbf{L}$ over $\mathbf{K}$, and suppose that D_f, the discriminant of the square-free monic polynomial divisor of maximal degree of f in $O_{\mathbf{K}}[X]$ is contained in S. Consider the solutions of equation (1) in $x \in O_{\mathbf{K}}$, $y \in O_{\mathbf{K}} \setminus \{0\}$,$w \in S$ and $n \geq 2$.

For $y \in O_{\mathbf{K}}^*$, n can be arbitrarily large. Further, if

$$f(X) = u^m f'(u^{-1}(X + a)) \tag{2}$$

for some $a \in O_{\mathbf{K}}$, $u \in S$, $f' \in O_{\mathbf{K}}[X]$ and $x + a = u = v^n$ with some $v \in S$, then $y = v^m$ yields a solution of (1), provided that $f'(1) \in S$. In this case $D_{f'} \in S$, $y \in S \setminus O_{\mathbf{K}}^*$ if $u \in S \setminus O_{\mathbf{K}}^*$, and n can be again arbitrarily large compared to m, k, Q and the parameters of $\mathbf{L}$. To exclude the above situation, in the case $y \in S \setminus O_{\mathbf{K}}^*$ we assume that f is *reduced*, that is that (2) does not hold for any a, f', u with $u \in S \setminus O_{\mathbf{K}}^*$.

Theorem 1. *Let $x, y \neq 0, w, n$ be a solution of (1) with $x \in O_{\mathbf{K}}$, $y \in O_{\mathbf{K}} \setminus O_{\mathbf{K}}^*$, $w \in S$, $n \geq 2$. If $y \notin S$ or if $y \in S$ and f is reduced, then*

$$n \leq c_1 Q^{c_2}, \tag{3}$$

where c_1, c_2 are effectively computable positive numbers which depend only on m, k, d and the discriminant $D_{\mathbf{L}}$ of $\mathbf{L}$.

We note that much better upper bounds come for n from our proof if y has a prime ideal divisor of large norm or if $y \in S$ and f is reduced. Further, in view of $D_f \in S$ we have

$$|D_{\mathbf{L}}| \leq c_3 Q^{c_4}, \tag{4}$$

where c_3, c_4 are effectively computable positive numbers depending only on k, d and the discriminant $D_{\mathbf{K}}$ of $\mathbf{K}$; for explicit values of c_3 and c_4, see Remark 5 after the proof of Theorem 2. Hence, together with (4), Theorem 1 provides also a bound for n which depends only on m, k, d, $D_{\mathbf{K}}$ and Q. These bounds can be compared with Theorem 2.2 of [14], where the bound depends also on D_f.

Shorey, van der Poorten, Tijdeman and Schinzel [20] generalized the above-mentioned result of [19] on equation (1) for the equation

$$F(x, z) = wy^n, \tag{5}$$

where $F \in \mathbf{Z}[X, Y]$ is a binary form with $F(1, 0) \neq 0$ and with at least two distinct linear factors over $\overline{\mathbf{Q}}$, subject to the conditions that $\gcd(x, z)$ is bounded and z, w are divisible by finitely many fixed primes only. In the *monic* case, when $F(1, 0) = 1$, this was extended in [21] to the number

field case. An explicit version has been recently given by HARISTOY [14], where the bound on n depends on the height of F.

We give now a generalization of Theorem 1 for equation (5). Let $F(X, Z)$ denote a monic binary form of degree m with coefficients in $O_{\mathbf{K}}$ such that $F(X, 1)$ has $k \geq 2$ distinct zeros and that D_F, the discriminant of the square-free polynomial divisor of maximal degree of $F(X, 1)$ in $O_{\mathbf{K}}[X]$ is contained in S. Let $\mathbf{L}$ be the splitting field of F over $\mathbf{K}$, and $D_{\mathbf{L}}$ the discriminant of $\mathbf{L}$. Consider the solutions of (5) in $x \in O_{\mathbf{K}}$, $y \in O_{\mathbf{K}} \setminus \{0\}$, $z, w \in S$, $n \geq 2$.

It suffices to deal with the case $y \notin O_{\mathbf{K}}^*$, since otherwise n can be arbitrarily large. If $F(1, 1) \in S$, then $x = z = v^n$, $y = v^m$ is a solution of (5) for every $v \in S$ and n, that is n cannot be bounded. Similarly, if

$$F(X, Z) = F'(X + aZ, uZ) \tag{6}$$

with $a \in O_{\mathbf{K}}$, $u \in S$, $F' \in O_{\mathbf{K}}[X, Z]$ and $F'(1, 1) \in S$, then $D_{F'} \in S$ and $z = 1$, $x + a = u = v^n$, $y = v^m$ is a solution of (5) for any $v \in S$, and n cannot be bounded above in terms of m, k, d, $D_{\mathbf{L}}$ and Q only.

Excluding these two cases, n can be estimated from above as in Theorem 1. We say that F is *reduced* if (6) does not hold for any a, F' and u with $u \in S \setminus O_{\mathbf{K}}^*$.

The following theorem contains Theorem 1 as a special case with the choice $z = 1$, $\mu = 0$.

Theorem 2. *There exist effectively computable positive numbers c_5, c_6 and c_7 which depend only on m, k, d and $D_{\mathbf{L}}$ such that if $x, y \neq 0$, z, w, n is a solution of (5) with $x \in O_{\mathbf{K}}$, $y \in O_{\mathbf{K}} \setminus O_{\mathbf{K}}^*$, $z \in S$, $w \in S$, $n \geq 2$, then*

$$n \leq c_5 Q^{c_6} + \mu c_7 \log Q, \tag{7}$$

where $\mu = 0$ if $y \notin S$, and

$$\operatorname{ord}_{\mathfrak{p}_i}((x, z)) \leq \mu \quad \text{for } i = 1, \ldots, s \tag{8}$$

if $y \in S \setminus O_{\mathbf{K}}^*$ and F is reduced.

The dependence on $D_{\mathbf{L}}$ in (7) can be eliminated again by means of (4).

Binomial Thue–Mahler equations. Consider now the equation

$$ax^n - by^n = c, \tag{9}$$

where a, b are fixed non-zero elements of $O_{\mathbf{K}}$, and $x, y \in O_{\mathbf{K}} \setminus \{0\}$, $c \in S$, $n \geq 3$ are unknowns. As is known, for $n = 0, 1, 2, \ldots$, $ax^n + by^n$ can be regarded as a special binary recurrence sequence. Several upper bounds have been derived for n in (9) in terms of a, b and S; see [21], [26], [8] and the references occurring there.

To prove our Theorem 2, we shall need the following extension in which a and b are also unknowns, taken from S. Denote by h and R the class number and regulator of $\mathbf{K}$, respectively.

Theorem 3. *Let $a, b, c \in S, x, y \in O_{\mathbf{K}} \setminus \{0\}$, $n \geq 3$ be a solution of (9), and suppose that at least one of x and y is not contained in $O_{\mathbf{K}}^*$. There exists an effectively computable positive constant c_8 which depends only on d, h and R such that*

$$n \leq c_8 Q^{3h} + \nu \log Q, \tag{10}$$

where $\nu = 0$ if $x \notin S$ or $y \notin S$, and

$$\operatorname{ord}_{\mathfrak{p}_i}((ax^n, by^n, c)) \leq \nu \quad \text{for } i = 1, \ldots, s \tag{11}$$

otherwise.

It is clear that in the case $x, y \in S$ the condition (11) is necessary.

We remark that in our proof much better upper bounds are obtained in the following special cases: $x, y \in S$ (cf. (45), (52)); x or y has a prime ideal divisor of large norm (see (40)); $c \in O_{\mathbf{K}}^*$ (see Remark 4 after the proof of Theorem 3). These better bounds enable one to improve the bounds of Theorems 1 and 2 in the corresponding special cases.

In the particular case $\mathbf{K} = \mathbf{Q}$ it follows from Theorem 3 that if a, b, x, y, n are non-zero rational integers with $|xy| > 1$, $n \geq 3$ such that xy has a prime factor which does not divide $ab(ax^n - by^n)$, then $Q(ax^n - by^n)$, the product of distinct prime factors of $ax^n - by^n$ satisfies

$$|ax^n - by^n| \geq Q(ax^n - by^n) \geq (c_9/Q(ab))n^{1/3}, \tag{12}$$

where $Q(ab)$ denotes the product of distinct prime factors of ab, and c_9 is an effectively computable absolute constant. Theorem 3 and (12) should be compared with Theorem 2 of Yu and Hung [26] where the dependence on n is better, but the lower bound obtained for $Q(ax^n - by^n)$ depends not only on $Q(ab)$ but also on a and b themselves.

Remark 1. We note that the above constants c_1 to c_9 can be easily expressed in explicit form by using the explicit estimates in our proofs as well as explicit versions of Lemmas 1 to 8.

Remark 2. In the proof of Theorem 3 we utilize among other things some new estimates of MATVEEV [16] and YU [25] on linear forms in logarithms of algebraic numbers and a recent bound of GYŐRY and YU [13] on the solutions of S-unit equations. Theorem 2 will be deduced from Theorem 3 with the help of a recent effective theorem of GYŐRY [12] concerning monic binary forms having discriminants contained in S. We remark that Theorems 1 and 2 can be proven, with other bounds, without the use of Theorem 3 as well. This will be the subject of a forthcoming paper.

2. Auxiliary results

To prove our theorems we need some lemmas. For a non-zero algebraic number α of degree l over $\mathbf{Q}$, whose minimal polynomial over $\mathbf{Z}$ is $a\prod_{i=1}^{l}(X-\alpha_i)$,

$$h(\alpha)=\frac{1}{l}\left(\log|a|+\sum_{i=1}^{l}\log\max(1,|\alpha_i|)\right)$$

denotes the absolute logarithmic height of α. For properties of this height, see e.g. [23].

Let again $\mathbf{K}$ denote an algebraic number field, $O_{\mathbf{K}}$ its ring of integers and $O_{\mathbf{K}}^*$ its unit group with the parameters d, h, R and $D_{\mathbf{K}}$ specified above. Let r denote the unit rank of $O_{\mathbf{K}}$, and set $\delta_d=2/(\log 3d)^3$ if $d\geq 2$ and $\delta_d=\log 2$ if $d=1$.

Lemma 1. *Suppose that $r\geq 1$. There exists a fundamental system $\varepsilon_1,\dots,\varepsilon_r$ of units in $\mathbf{K}$ such that*

$$h(\varepsilon_i)\leq c_{10}R \quad (1\leq i\leq r) \quad \text{and} \quad \left|\varepsilon_i{}^{(j)}\right|\geq\exp\{-dc_{10}R\}$$

for each field conjugate $\varepsilon_i{}^{(j)}$ of ε_i, $1\leq i\leq r$, $1\leq j\leq d$, and

$$\prod_{i=1}^{r}h(\varepsilon_i)\leq c_{11}R,$$

where

$$c_{10} = \frac{(r!)^2}{2^{r-1}d^r}\left(\frac{\delta_d}{d}\right)^{1-r} \quad \textit{and} \quad c_{11} = c_{10}\left(\frac{d}{\delta_d}\right)^{1-r}.$$

PROOF. The first and third estimates are consequences of Lemma 1 of [7], the second one is an easy consequence of the first inequality. □

Lemma 2. *For every $\alpha \in O_{\mathbf{K}} \setminus \{0\}$ there exists a unit $\varepsilon \in O_{\mathbf{K}}$ such that*

$$h(\varepsilon\alpha) \le \frac{\log|N_{\mathbf{K}/\mathbf{Q}}(\alpha)|}{d} + c_{12}R,$$

and

$$\left|(\varepsilon\alpha)^{(j)}\right| \ge |N_{\mathbf{K}/\mathbf{Q}}(\alpha)|^{1/d}\exp\{-c_{12}R\}$$

for each field conjugate $(\varepsilon\alpha)^{(j)}$ of $\varepsilon\alpha$, $j = 1, \dots, d$, where $c_{12} = r^{r+1}\delta_d^{-(r-1)}/2$.

PROOF. The first inequality is a special case of Lemma 2 in [7], while the second one is an immediate consequence of (15) in the proof of Lemma 2 of [6]. □

Lemma 3. *If $d \ge 2$, then*

$$Rh < c_{13}|D_{\mathbf{K}}|^{1/2}(\log|D_{\mathbf{K}}|)^{d-1} \quad \textit{and} \quad R \ge 0.2052,$$

where $c_{13} = 2d^d/((2\pi)^{d/2}d!)$.

PROOF. For the first inequality, see [15]; the second one is proved in [9]. □

Let $\alpha_1, \dots, \alpha_m$ be non-zero elements of $\mathbf{K}$, and let $b_1, \dots, b_m$ be rational integers with $B = \max(|b_1|, \dots, |b_m|, 3)$. Put

$$\Lambda = {\alpha_1}^{b_1} \cdots {\alpha_m}^{b_m} - 1.$$

Lemma 4. *Let*

$$A_j \ge \max\{dh(\alpha_j), |\log\alpha_j|, 0.16\} \quad (1 \le j \le m).$$

If $\Lambda \ne 0$ then

$$\log|\Lambda| > -C_1(m,d)\log(meB)A_1A_2\cdots A_m,$$

where

$$C_1(m,d) = 3.15 \cdot 30^{m+4}(m+1)^{5.5}d^2\log(ed).$$

PROOF. As is known, if $0 < |\Lambda| < 1/3$ then

$$|b_0 \log(-1) + b_1 \log \alpha_1 + \cdots + b_m \log \alpha_m| \leq 2|\Lambda|,$$

where $b_0 \in \mathbf{Z}$ with $|b_0| \leq mB$ and log denotes the principal value of the logarithm. Hence Lemma 4 is an immediate consequence of Corollary 2.3 of [16]. □

Remark 3. Set $c_{14} = 3.22d/\delta_d$. It is easy to check that in Lemma 4, we can choose $A_j = \pi$ or $A_j = c_{14}h(\alpha_j)$ according as α_j is a root of unity or not.

Denote by $\mathfrak{p}$ a prime ideal of $O_{\mathbf{K}}$ lying above the prime number p, and by $e_{\mathfrak{p}}$ the ramification index of $\mathfrak{p}$. For $\alpha \in \mathbf{K}^*$, write $\operatorname{ord}_{\mathfrak{p}} \alpha$ for the exponent to which $\mathfrak{p}$ divides the fractional ideal (α) generated by α in $\mathbf{K}$.

Lemma 5. *If $\alpha \in \mathbf{K}^*$ then*

$$\operatorname{ord}_{\mathfrak{p}} \alpha \leq \frac{d}{\log N_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p})} h(\alpha).$$

PROOF. See YU [24], p. 124. □

The following lemma is a consequence of Theorem 3 in YU [25]. It is a p-adic analogue of Lemma 4.

Lemma 6. *Let again $\alpha_1, \ldots, \alpha_m$ be non-zero elements of $\mathbf{K}$, and suppose that they are not roots of unity. If $\Lambda \neq 0$ then*

$$\operatorname{ord}_{\mathfrak{p}} \Lambda < C_2(m, d, \mathfrak{p}) h(\alpha_1) \cdots h(\alpha_m) \log B,$$

where

$$C_2(m, d, \mathfrak{p}) = (16ed)^{2(m+1)} m^{5/2} {e_{\mathfrak{p}}}^m \frac{N_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p})}{\log^2 N_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p})} \log(2md) \log(2d).$$

PROOF. This is a consequence of Theorem 3 in [25]. □

As in Section 1, $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ will denote distinct prime ideals in $O_{\mathbf{K}}$. In what follows, we use the notation

$$P = \begin{cases} \max_{1 \leq i \leq s} N_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p}_i) & \text{if } s > 0, \\ 1 & \text{if } s = 0. \end{cases}$$

Further, let

$$W = (\log^* N_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p}_1)) \cdots (\log^* N_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p}_s)),$$

if $s > 0$ and $W = 1$ if $s = 0$. Here $\log^* a$ is defined as $\max\{\log a, 1\}$.

The element $\alpha \in \mathbf{K}$ is called S-unit if $\mathrm{ord}_{\mathfrak{p}}(\alpha) = 0$ for every prime ideal $\mathfrak{p}$ different from $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$. Let O_S^* denote the group of S-units in $\mathbf{K}$. Clearly $S = O_S^* \cap O_{\mathbf{K}}$. Consider the S-unit equation

$$x_1 + x_2 + x_3 = 0 \quad \text{in} \quad x_1, x_2, x_3 \in O_S^*. \tag{13}$$

Put

$$T = \prod_{i=1}^{s} \max\{h \log p_i, c_{12}R\} \text{ and } H = \max\{h, c_{12}R\},$$

where $p_1, \ldots, p_s$ denote the rational primes lying below $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$, respectively.

Lemma 7. *For every solution x_1, x_2, x_3 of (13) there are $\sigma \in O_S^*$ and $\rho_k \in S$ such that*

$$x_k = \sigma\rho_k, \quad k = 1, 2, 3$$

and

$$\max_{1 \le k \le 3} h(\rho_k) \le c_{15} h R^2 (\log^* R) H^2 P T \log T,$$

where $c_{15} = (13d)^{3(2r+s+7)}$. Further, if $x_k \in S$ for $k = 1, 2, 3$, then σ can be chosen from S.

Proof. This is a consequence of Theorem 2 of [13]. □

For a polynomial F with algebraic coefficients, we denote by $h(F)$ the maximum of the heights of the coefficients of F.

Lemma 8. *Let $F \in O_{\mathbf{K}}[X, Y]$ be a monic binary form of degree $k \ge 3$ with discriminant $D(F) \in S$ and splitting field $\mathbf{L}$ over $\mathbf{K}$. Then there are a monic $F^* \in O_{\mathbf{K}}[X, Y]$ and $a, u \in O_{\mathbf{K}}$ with $u \in S$ such that*

$$F(X, Y) = F^*(X + aY, uY)$$

and

$$h(F^*) \le c_{16} c_{17}^s P^l W^{l+1},$$

where $l = [\mathbf{L} : \mathbf{K}]$ *and* c_{16}, c_{17} *are effectively computable positive numbers which depend on* d, k *and the discriminant* $D_{\mathbf{L}}$ *of* $\mathbf{L}$.

PROOF. This is a special case of Corollary 2 in [12]. □

Finally, we shall need the concept of the S-norm. If $\alpha \in \mathbf{K}$ then (α) can be written uniquely as a product of two ideals $\mathfrak{a}_1, \mathfrak{a}_2$ where $\mathfrak{a}_1$ is composed of $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ and $\mathfrak{a}_2$ is composed solely of prime ideals different from $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$. Then the S-norm of α is defined as $N_S(\alpha) = N_{\mathbf{K}/\mathbf{Q}}(\mathfrak{a}_2)$. It is clear that $\alpha \in O_{\mathbf{K}}$ is contained in S if and only if $N_S(\alpha) = 1$.

3. Proofs of the theorems

We first prove Theorem 3.

PROOF OF THEOREM 3. In our proof we shall use some ideas from [26] and [8], where equation (9) is studied with fixed a, b. We keep the notation of the preceding sections. If $r + s = 0$, then (10) follows immediately with $\nu = 0$ from a classical theorem of ZSIGMONDY [27]. Thus we may assume that $r + s > 0$. For $s > 0$, we can write

$$(a) = \prod_{i=1}^{s} \mathfrak{p}_i^{a_i}, \quad (b) = \prod_{i=1}^{s} \mathfrak{p}_i^{b_i}$$

where a_i, b_i are non-negative integers. Let h_i denote the smallest positive integer for which $\mathfrak{p}_i^{h_i}$ is a principal ideal. Put $a_i = c_i h_i + d_i$, $b_i = e_i h_i + f_i$ with integers $c_i, d_i, e_i, f_i \geq 0$ such that

$$0 \leq d_i, f_i < h_i$$

for $i = 1, 2, \ldots, s$. We infer that

$$a = \varepsilon_a \alpha \prod_{i=1}^{s} \pi_i^{c_i} \quad b = \varepsilon_b \beta \prod_{i=1}^{s} \pi_i^{e_i}, \tag{14}$$

where ε_a, ε_b are units and π_i, α, β are integers in $O_{\mathbf{K}}$ such that

$$(\pi_i) = \mathfrak{p}_i^{h_i}, \ (\alpha) = \prod_{i=1}^{s} \mathfrak{p}_i^{d_i}, \ (\beta) = \prod_{i=1}^{s} \mathfrak{p}_i^{f_i}.$$

On applying Lemma 2 we deduce that in (14) α and β can be chosen so that

$$\min\{|\alpha^{(j)}|, |\beta^{(j)}|\} \geq \exp\{-c_{12}R\} \tag{15}$$

for each j with $1 \leq j \leq d$. Further, we have

$$h(\alpha) \leq \frac{\log |N_{\mathbf{K}/\mathbf{Q}}(\alpha)|}{d} + c_{12}R \quad \text{and} \quad h(\beta) \leq \frac{\log |N_{\mathbf{K}/\mathbf{Q}}(\beta)|}{d} + c_{12}R,$$

whence

$$\max(h(\alpha), h(\beta)) \leq (sh/d)\log P + c_{12}R = c_{16}. \tag{16}$$

Similarly, for $s > 0$, we can choose π_i such that

$$|\pi_i{}^{(j)}| \geq \exp\{-c_{12}R\} \quad \text{for} \quad j = 1, \ldots, d \tag{17}$$

and

$$h(\pi_i) \leq \frac{h_i \log N_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p}_i)}{d} + c_{12}R, \tag{18}$$

whence

$$\prod_{i=1}^{s} h(\pi_i) \leq \prod_{i=1}^{s}((h_i/d)\log N_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p}_i) + c_{12}R) = c_{17}. \tag{19}$$

We recall that $c_{12} = 0$ for $r = 0$. If $r \geq 1$, fix a fundamental system $\varepsilon_1, \ldots, \varepsilon_r$ of units in $\mathbf{K}$ with the properties specified in Lemma 1. Then, incorporating the roots of unity from the representation of $\varepsilon_a, \varepsilon_b$ into α, β, we can rewrite (9) as

$$\alpha \prod_{j=1}^{r} \varepsilon_j^{u_j} \prod_{i=1}^{s} \pi_i^{t_i} (x')^n - \beta \prod_{j=1}^{r} \varepsilon_j^{v_j} \prod_{i=1}^{s} \pi_i^{w_i} (y')^n = c \tag{20}$$

where we may assume that t_i, w_i, u_j, v_j are rational integers which satisfy

$$0 \leq u_j, v_j, t_i, w_i < n \tag{21}$$

for each j and i, and where x', y' denote non-zero integers in $\mathbf{K}$ such that

$$\max(N_S(x'), N_S(y')) = \max(N_S(x), N_S(y)).$$

Hence, for simplicity, we may write in (20) x, y in place of x', y'. Set

$$R_n = ax^n - by^n,$$

where now

$$a = \alpha \prod_{j=1}^{r} \varepsilon_j^{u_j} \prod_{i=1}^{s} \pi_i^{t_i} \quad \text{and} \quad b = \beta \prod_{j=1}^{r} \varepsilon_j^{v_j} \prod_{i=1}^{s} \pi_i^{w_i}. \tag{22}$$

It follows from (20) that $((x)^h, (y)^h) = (\omega)$ with some $\omega \in S$. Put

$$x_\omega = \frac{x^h}{\omega}, \ y_\omega = \frac{y^h}{\omega}. \tag{23}$$

It is clear that x_ω, y_ω are integers in $\mathbf{K}$. Denote by $x_\omega^{(j)}, y_\omega^{(j)}$, $j = 1, \dots, d$, their field conjugates. Let

$$M_\omega = \prod_{j=1}^{d} \max(|x_\omega^{(j)}|, |y_\omega^{(j)}|), \tag{24}$$

and

$$S_n = \frac{R_n^h}{\omega^n}.$$

It is easy to see that $S_n \in S$. A straightforward calculation gives

$$S_n = b^h y_\omega^n \left(\frac{a}{b} \left(\frac{x}{y} \right)^n - 1 \right)^h = a^h x_\omega^n \left(\frac{b}{a} \left(\frac{y}{x} \right)^n - 1 \right)^h. \tag{25}$$

We distinguish two cases. First suppose that

$$|N_{\mathbf{K}/\mathbf{Q}}(S_n)| \le M_\omega^{0.99n}. \tag{26}$$

We note that this holds if e.g. $s = 0$ or $s > 0$ but $c \in O_{\mathbf{K}}^*$. We are going to derive a lower bound for $|N_{\mathbf{K}/\mathbf{Q}}(S_n)|$. We may suppose that $|y_\omega| \ge |x_\omega|$. Then we deduce from (25) that

$$|S_n| \ge (\min(|a|, |b|))^h (\max(|x_\omega|, |y_\omega|))^n \times \left| \left(\frac{\alpha}{\beta} \right) \prod_{j=1}^{r} \varepsilon_j^{u_j - v_j} \prod_{i=1}^{s} \pi_i^{t_i - w_i} \left(\frac{x}{y} \right)^n - 1 \right|^h.$$

First we give a lower bound for $\min(|a|, |b|)$. Combining (22) with (15), (17) and Lemma 1, we deduce that

$$|a| = |\alpha| \prod_{j=1}^{r} |\varepsilon_j|^{u_j} \prod_{i=1}^{s} |\pi_i|^{t_i} \ge \exp\{-n(r+s+1)dc_0R\}$$

with $c_0 = \max\{c_{12}, c_{10}\}$, and the same lower bound follows for $|b|$. Here $c_0 \geq 1/2$ if $r \geq 1$, and we put $c_0 = 0$ if $r = 0$. We infer that

$$\min(|a|, |b|) \geq \exp\{-nc_{18}\},$$

where $c_{18} = (r + s + 1)dc_0R$. Put

$$\Lambda_1 = \left(\frac{\alpha}{\beta}\right) \prod_{j=1}^{r} \varepsilon_j^{u_j - v_j} \prod_{i=1}^{s} \pi_i^{t_i - w_i} \left(\frac{x}{y}\right)^n - 1. \tag{27}$$

In view of (9) and (22) we have $\Lambda_1 \neq 0$. We shall now apply Lemma 4 to derive a lower bound for $|\Lambda_1|$. We may assume that x/y is not a root of unity. Otherwise (9) and (11) imply at once that $n \leq \nu$. Then, as was showed in Yu and Hung ([26], pp. 352–353), $h(x/y) \leq \log M_\omega$ holds. Further, we may assume that $n > (2e(r + s + 2))^{10}$, and hence that $\log(2n) \leq 1.1n$. Then Lemma 4 yields

$$|\Lambda_1| \geq \exp\{-c_{19} \log M_\omega \log n\},$$

where

$$c_{19} = 2.2C_1(r + s + 2, d)(c_{11}R)c_{14}^{r+s+2}c_{16}c_{17},$$

with the constants C_1, c_{11}, c_{14}, c_{16} and c_{17} specified above. For $r = 0$, we may write 1 for $c_{11}R$. Repeating the above argument for every conjugate of S_n and taking the product of the inequalities so obtained, we get

$$|N_{\mathbf{K}/\mathbf{Q}}(S_n)| \geq \exp\{-nhdc_{18}\} M_\omega^n \exp\{-dhc_{19} \log M_\omega \log n\}. \tag{28}$$

Putting $c_{20} = \exp\{dc_{18}\}$ and comparing (26) and (28), we infer that

$$n(0.01 \log M_\omega - h \log c_{20}) \leq dhc_{19} \log M_\omega \log n. \tag{29}$$

We note that $c_{20} > 1$ if $r \geq 1$, and $c_{20} = 1$ if $r = 0$. Suppose that

$$\max(N_S(x), N_S(y)) > c_{20}^{200}. \tag{30}$$

Using (23) and (24), it is not difficult to show that

$$(\max(N_S(x), N_S(y)))^h \leq \max(|N_{\mathbf{K}/\mathbf{Q}}(x_\omega)|, |N_{\mathbf{K}/\mathbf{Q}}(y_\omega)|) \leq M_\omega. \tag{31}$$

Hence (29) and (30) give

$$n \leq 200dhc_{19} \log n,$$

whence

$$n \leq 400dhc_{19} \log (200dhc_{19})$$

follows. This yields

$$n \leq c_{21} c_{22}^s W(\log P), \tag{32}$$

where c_{21} and c_{22} depend only on d, h and R and can be easily evaluated in explicit form from c_{19}. This implies (10) with $\nu = 0$ under the assumption (30).

It remains the case

$$|N_{\mathbf{K}/\mathbf{Q}}(S_n)| > M_\omega^{0.99n}. \tag{33}$$

Then obviously $s > 0$ holds. We give an upper bound for $|N_{\mathbf{K}/\mathbf{Q}}(S_n)|$. Fix an index i with $1 \leq i \leq s$. Since $(x_\omega, y_\omega) = 1$, we may assume that $\mathrm{ord}_{\mathfrak{p}_i}(y_\omega) = 0$. We deduce from (25) and (27) that

$$\mathrm{ord}_{\mathfrak{p}_i}(S_n) = h \,\mathrm{ord}_{\mathfrak{p}_i}(b) + h \,\mathrm{ord}_{\mathfrak{p}_i}(\Lambda_1). \tag{34}$$

Lemma 5 and (22) yield

$$\mathrm{ord}_{\mathfrak{p}_i}(b) \leq hw_i + \frac{d}{\log N_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p}_i)} h(\beta) < hn + c_{23}, \tag{35}$$

where $c_{23} = dc_{16}/\log N_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p}_i)$. On applying Lemma 6, we now give an upper bound for $\mathrm{ord}_{\mathfrak{p}_i}(\Lambda_1)$. Denote by m the number of roots of unity in $\mathbf{K}$. Then $\phi(m) \mid d$, and hence $m \leq 20d \log\log d$ (cf. [18]), where $\phi(m)$ denotes Euler's function. We may assume that $\log(2mn) \leq 2n$ and that $\mathrm{ord}_{\mathfrak{p}_i} \Lambda_1 > m$. Together with (22), (16), (19), Lemma 1 and Lemma 6 give

$$\begin{aligned} \mathrm{ord}_{\mathfrak{p}_i} \Lambda_1 &< \mathrm{ord}_{\mathfrak{p}_i} \left(\left(\left(\frac{\alpha}{\beta} \right) \prod_{j=1}^{r} {\varepsilon_j}^{u_j - v_j} \prod_{i=1}^{s} {\pi_i}^{t_i - w_i} \left(\frac{x}{y} \right)^n \right)^m - 1 \right) \\ &\leq c_{24} \log M_\omega \log n, \end{aligned} \tag{36}$$

where

$$c_{24} = 4(\log 7d) C_2(r + s + 2, d, \mathfrak{p}_i)(c_{11}R) c_{16} c_{17}$$

with the constants $C_2, c_{11}, c_{16}, c_{17}$ specified above. For $r = 0$, we may write again 1 in place of $c_{11}R$. Thus (34), (35), (36) and $c_{24} > c_{23}$ imply that

$$\mathrm{ord}_{\mathfrak{p}_i}(S_n) < h^2 n + 2hc_{24} \log M_\omega \log n,$$

whence we deduce that

$$\begin{aligned}\log|N_{\mathbf{K}/\mathbf{Q}}(S_n)| &= \sum_{i=1}^{s} \mathrm{ord}_{\mathfrak{p}_i}(S_n)\log N_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p}_i)\\ &\le n\log c_{25} + c_{26}\log M_\omega \log n,\end{aligned} \tag{37}$$

where $c_{25} = Q^{h^2}$ and $c_{26} = 2h(\log Q)c_{24}$. Then comparing (33) and (37), we infer that

$$n(0.99\log M_\omega - \log c_{25}) \le c_{26}\log M_\omega \log n. \tag{38}$$

If now

$$\max(N_S(x), N_S(y)) > Q^{1.02h} \tag{39}$$

then, by (31), $\log M_\omega > 1.02\log c_{25}$. Hence we deduce from (38) that

$$n \le 400c_{26}\log(200c_{26}).$$

Thus we proved that if (30) and (39) hold then

$$n \le 400\max\{dhc_{19}\log(200dhc_{19}), c_{26}\log(200c_{26})\}.$$

We may assume that

$$\log Q > 200(r+s+1)dRc_0,$$

since otherwise we obtain a better bound for n in terms of S. Then we have $Q^{1.02h} > c_{20}^{200}$. Using the fact that $\log Q \le s\log P$, it follows that if (39) holds then

$$n \le c_{27}c_{28}^{s}PW(\log^* W), \tag{40}$$

where c_{27}, c_{28} depend only on d, h and R, and it is easy to evaluate them in explicit form. By (39), we have x or $y \notin S$, and (40) gives at once (10) with $\nu = 0$.

In what follows, we consider the case

$$\max(N_S(x), N_S(y)) \le C_3, \tag{41}$$

where

$$C_3 = \begin{cases} Q^{1.02h} & \text{if } s > 0,\\ c_{20}^{200} & \text{if } s = 0.\end{cases}$$

We recall that $c_{20} = \exp\{(r+1)d^2c_0R\} > 1$ for $s = 0$. First assume that x or y is not contained in S. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_t$ be the distinct prime ideal divisors of xy which are different from $\mathfrak{p}_1, \dots, \mathfrak{p}_s$. If P_q and Q_q denote the greatest and the product of the norms of the prime ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_t$, then (41) gives

$$P_q \le Q_q \le N_S(xy) \le {C_3}^2. \tag{42}$$

Further it follows from explicit estimates of Rosser and Schoenfeld [18] concerning primes that

$$t \le 3\frac{\log Q_q}{\log_2 Q_q},$$

where $\log_i$ denotes the i-fold iterated logarithm; cf. e.g. [14]. It suffices to deal with the case $Q_q \ge Q$ if $s > 0$ and $Q_q \ge c_{20}^{200}$ if $s = 0$. Hence we infer that

$$t \le \begin{cases} 6.12h(\log Q/\log_2 Q) & \text{for } s > 0, \\ 1200(\log c_{20}/\log(200\log c_{20})) & \text{for } s = 0. \end{cases} \tag{43}$$

Similarly, $s \le 3\frac{\log Q}{\log_2 Q}$. Denote by S' the set of prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ $\mathfrak{q}_1, \dots, \mathfrak{q}_t$. Then (9) can be regarded as an S'-unit equation with $ax^n, by^n, c \in O_{S'}^* \cap O_{\mathbf{K}}$. Hence, by Lemma 7, it follows that

$$ax^n = \sigma\rho_1, \quad by^n = \sigma\rho_2, \quad c = \sigma\rho_3, \tag{44}$$

where

$$\sigma \in O_{S'}^* \cap O_{\mathbf{K}} \quad \text{and} \quad \rho_k \in O_{S'}^* \cap O_{\mathbf{K}}$$

such that

$$\max_{1\le k\le 3} h(\rho_k) \le c_{29}hR^2(\log^* R)H^2P'T'(\log T'). \tag{45}$$

Here $c_{29} = (13d)^{3(2r+s+t+7)}$, $P' = \max\{P, P_q\}$, $H = \max\{h, c_{12}R\}$ and

$$T' = \prod_{i=1}^{s} \max\{h_i \log p_i, c_{12}R\} \prod_{j=1}^{t} \max\{h \log q_j, c_{12}R\}, \tag{46}$$

where $q_1, \dots, q_t$ denote the rational primes lying below $\mathfrak{q}_1, \dots, \mathfrak{q}_t$, respectively. But

$$\begin{cases} \prod_{i=1}^{s} \max\{h_i \log p_i, c_{12}R\} \le ((c_{12}+5)hR)^s \prod_{i=1}^{s} (\log^* p_i) \\ \text{and} \\ \prod_{j=1}^{t} \max\{h \log q_j, c_{12}R\} \le ((c_{12}+5)hR)^t \prod_{j=1}^{t} (\log^* q_j). \end{cases} \tag{47}$$

Further, for $s > 0$ it is easy to deduce from explicit estimates of [18] (see e.g. [14]) that

$$\prod_{i=1}^{s} \log^* p_i \le Q^{3(\log d + 2\log_3 Q)/\log_2 Q} \tag{48}$$

and

$$\begin{aligned} \prod_{j=1}^{t} \log^* q_j &\le {Q_q}^{3(\log d + 2\log_3 Q_q)/\log_2 Q_q} \\ &\le \begin{cases} Q^{6.12h(\log d + 2\log_2(3h\log Q))/\log_2 Q} & \text{if } s > 0, \\ c_{20}^{1200(\log d + 2\log_2(400\log c_{20}))/\log_2 c_{20}} & \text{if } s = 0. \end{cases} \end{aligned} \tag{49}$$

By virtue of $c \in S$ and $c = \sigma\rho_3$, we obtain that $\sigma \in S$. Thus $\mathfrak{q}_1 \cdots \mathfrak{q}_t \mid xy$ and (44) imply that $(\mathfrak{q}_1 \cdots \mathfrak{q}_t)^n \mid \rho_1\rho_2$, whence

$$n \log N_{\mathbf{K}/\mathbf{Q}}(\mathfrak{q}_1 \cdots \mathfrak{q}_t) \le \log |N_{\mathbf{K}/\mathbf{Q}}(\rho_1\rho_2)| \le d(h(\rho_1) + h(\rho_2)). \tag{50}$$

If $s > 0$, we may suppose that

$$\log Q/(\log_2 Q)^{50} > 13d^2hR\log^{25}(3h)(c_{12}+5)$$

since otherwise we get a better bound for n. Now (42)–(50) give

$$n \le c_{30}Q^{3h}, \tag{51}$$

where c_{30} depends only on d, h and R, and, using (42)–(50), it can be easily given in explicit form. Further, c_{30} can be chosen so that, for $s > 0$, this bound in (51) is larger than that in (40). This proves (10) with $\nu = 0$ for the case when $x \notin S$ or $y \notin S$.

Consider now the case when, in (9), $s > 0, x, y \in S$ and (11) holds, but say x, is not contained in $O_{\mathbf{K}}^*$. Then, by Lemma 7, we obtain again

(44) where now $\sigma \in S$, $\rho_k \in S$ and (45) holds with $c_{29} = (13d)^{3(2r+s+7)}$, $P' = P$ and

$$T' = \prod_{i=1}^{s} \max\{h \log p_i, c_{12}R\}.$$

(11) and (44) imply that

$$\operatorname{ord}_{\mathfrak{p}_i} \sigma \leq \nu \quad \text{for} \quad i = 1, \dots s,$$

whence, in view of $x \in S \setminus O_{\mathbf{K}}^*$, it follows that

$$\begin{aligned} n \log |N_{\mathbf{K}/\mathbf{Q}}(x)| &\leq \log |N_{\mathbf{K}/\mathbf{Q}}(\sigma\rho_1)| \\ &= \log |N_{\mathbf{K}/\mathbf{Q}}(\sigma)| + \log |N_{\mathbf{K}/\mathbf{Q}}(\rho_1)| \\ &\leq \nu \log Q + dh(\rho_1). \end{aligned} \tag{52}$$

Together with (45), this implies (10) with an appropriate constant c_8 which depends only on d, h and R and which can be given explicitly. This completes the proof of Theorem 3. □

Remark 4. If in particular $c \in O_{\mathbf{K}}^*$ (e.g. if $s = 0$), then in our proof $w, S_n \in O_{\mathbf{K}}^*$, hence (33) cannot hold. In this case our proof provides a much better bound for n.

Theorem 2 will be deduced from Theorem 3 by means of Lemma 8.

Proof of Theorem 2. Putting

$$F(X, Z) = (X - \alpha_1 Z)^{a_1} \cdots (X - \alpha_k Z)^{a_k}$$

with distinct $\alpha_1, \dots, \alpha_k$ contained in $O_{\mathbf{L}}$, the ring of integers of $\mathbf{L}$, we have

$$D_F = \prod_{1 \leq i < j \leq k} (\alpha_i - \alpha_j)^2.$$

Let $x \in O_{\mathbf{K}}$, $y \in O_{\mathbf{K}} \setminus \{0\}$, $z, w \in S$ and $n \geq 3$ be a solution of (5) with $y \notin O_{\mathbf{K}}^*$. We can deduce from (5) in a standard way that

$$(x - \alpha_i z) = \mathfrak{B}_i \mathfrak{b}_i^{n'}, \quad i = 1, \dots, k, \tag{53}$$

where

$$n' = \frac{n}{\gcd(n, \operatorname{lcm}(a_1, \dots, a_k))}$$

and $\mathfrak{B}_i$, $\mathfrak{b}_i$ are non-zero ideals in $O_{\mathbf{L}}$ such that $\mathfrak{B}_i$ is composed only of prime ideals dividing $\mathfrak{p}_1 \cdots \mathfrak{p}_s$. Let $\mathbf{M}$ denote the Hilbert class field of $\mathbf{L}$, and $R_{\mathbf{M}}, h_{\mathbf{M}}$ and $D_{\mathbf{M}}$ its regulator, class number and discriminant, respectively. Denote by $S_{\mathbf{M}}$ the set of those elements of $O_{\mathbf{M}}$, the ring of integers of $\mathbf{M}$, whose all prime ideal divisors divide $\mathfrak{p}_1 \cdots \mathfrak{p}_s$. Then (53) implies that

$$x - \alpha_i z = \beta_i \gamma_i^{n'}, \quad i = 1, 2, \tag{54}$$

where $\beta_i \in S_{\mathbf{M}}$, $\gamma_i \in O_{\mathbf{M}} \setminus \{0\}$. Put $l = [\mathbf{L} : \mathbf{K}]$. Since $[\mathbf{M} : \mathbf{L}]$ is equal to $h_{\mathbf{L}}$, the class number of $\mathbf{L}$, the number of prime ideals of $O_{\mathbf{M}}$ which divide $\mathfrak{p}_1 \cdots \mathfrak{p}_s$ is at most $slh_{\mathbf{L}}$. But (54) gives

$$\beta_1 \gamma_1^{n'} - \beta_2 \gamma_2^{n'} = (\alpha_2 - \alpha_1) z, \tag{55}$$

where, in view of $D_F \in S$, $(\alpha_2 - \alpha_1)z \in S_{\mathbf{M}}$ holds.

First consider the case when $y \notin S$. Then for at least one i, say for $i = 1$, $\mathfrak{b}_i$ has a prime ideal divisor in $O_{\mathbf{M}}$ which is relatively prime to $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$. Thus $\gamma_1 \notin S_{\mathbf{M}}$. On applying now our Theorem 3 to (55), we deduce that

$$n' \le c_{31} Q^{h_{\mathbf{M}} l h_{\mathbf{L}}},$$

whence

$$n \le c_{31} \mathrm{lcm}(a_1, \ldots, a_k) Q^{c_{32}}, \tag{56}$$

where c_{31}, c_{32} are effectively computable positive numbers depending only on l, $h_{\mathbf{L}}$, $[\mathbf{M} : \mathbf{Q}] = m$, $R_{\mathbf{M}}$ and $h_{\mathbf{M}}$. But, by Lemma 3, $R_{\mathbf{M}}$ and $h_{\mathbf{M}}$ can be bounded above by an explicit expression of m and $D_{\mathbf{M}}$. Further, $m = dlh_{\mathbf{L}}$ and, as is known, $D_{\mathbf{M}} = {D_{\mathbf{L}}}^{h_{\mathbf{L}}}$. Hence $m, R_{\mathbf{M}}$ and $h_{\mathbf{M}}$ can be explicitly estimated from above in terms of d, l, $D_{\mathbf{L}}$ and $h_{\mathbf{L}}$. Finally, using again Lemma 3, $h_{\mathbf{L}}$ can also be estimated from above in terms of d, l and $D_{\mathbf{L}}$. But $l \le k!$, thus (56) yields (7) with the choice $\mu = 0$.

Consider now the case when $y \in S \setminus O_{\mathbf{K}}^*$. Then, by assumption, F is reduced. Hence Lemma 8 implies that there are a monic binary form F' with coefficients in $O_{\mathbf{K}}$ and $a \in O_{\mathbf{K}}, u \in O_{\mathbf{K}}^*$ such that

$$F(x, z) = F'(x', z') \tag{57}$$

with

$$x' = x + az, z' = uz \tag{58}$$

and

$$h(F') \le c_{33}c_{34}^{s}P^{l}W^{l+1} := C_4,$$

where c_{33}, c_{34} are effectively computable positive numbers which depend only on d, k and $D_{\mathbf{L}}$. This implies that if $\alpha_1', \dots, \alpha_k'$ denote the zeros of the polynomial $F'(X', 1)$ then

$$h(\alpha_i' - \alpha_j') \le dC_4 + \log k \quad \text{for each } i \text{ and } j. \tag{59}$$

Further, in view of (58) and the assumption (8), we have

$$\mathrm{ord}_{\mathfrak{p}_i}((x', z')) \le \mu \quad \text{for each } i.$$

We should now consider the equation $F'(x', z') = wy^n$ in place of (5). For simplicity, we write F, x, z and α_i instead of F', x', z' and α_i'. Then we obtain again (54) and (55). Further, $\gamma_i \in S_{\mathbf{M}}$ for each i, and $y \in S \setminus O_{\mathbf{K}}^*$ implies that there is a γ_i, say γ_1, which is not contained in $O_{\mathbf{M}}^*$, the unit group of $O_{\mathbf{M}}$. Let $\mathfrak{P}$ be any prime ideal divisor of $\mathfrak{p}_1 \cdots \mathfrak{p}_s$ in $O_{\mathbf{M}}$. Then, by (59) we have

$$\begin{aligned} \mathrm{ord}_{\mathfrak{P}}(\alpha_1 - \alpha_2) \log N_{\mathbf{M/Q}}(\mathfrak{P}) &\le \log |N_{\mathbf{M/Q}}(\alpha_1 - \alpha_2)| \\ &\le dlh_{\mathbf{L}} h(\alpha_1 - \alpha_2) \\ &\le dlh_{\mathbf{L}}(dC_4 + \log k). \end{aligned}$$

Further, it follows that

$$\begin{aligned} \mathrm{ord}_{\mathfrak{P}}(\beta_1\gamma_1^{n'}, \beta_2\gamma_2^{n'}, (\alpha_2 - \alpha_1)z) &= \mathrm{ord}_{\mathfrak{P}}(x - \alpha_1 z, x - \alpha_2 z, (\alpha_2 - \alpha_1)z) \\ &\le \mathrm{ord}_{\mathfrak{P}}((\alpha_2 - \alpha_1)(x, z)) \le dlh_{\mathbf{L}}(2dC_4 + 2\log k + \mu). \end{aligned}$$

We can now apply again Theorem 3 to (55), and we deduce as above that there are effectively computable positive numbers c_{35}, c_{36} and c_{37} depending only on d, k and $D_{\mathbf{L}}$, such that

$$n' \le c_{35}Q^{c_{36}} + \mu c_{37} \log Q$$

holds, whence (7) follows. □

Remark 5. It follows from (21) in [10] and the proof of Corollary 5 in [11] that

$$|D_{\mathbf{L}}| \le \left(|D_{\mathbf{K}}|^{k} Q^{dk-1} e^{s(dk)^2}\right)^{dl}, \tag{60}$$

where $l = [\mathbf{L} : \mathbf{K}] \leq k!$. Further, as was seen above, $s \leq 3\frac{\log Q}{\log_2 Q}$. Thus (60) implies (4) in Section 1.

References

[1] A. Bérczes, B. Brindza and L. Hajdu, On power values of polynomials, *Publ. Math. Debrecen* **53** (1998), 375–381.

[2] B. Brindza, Zeros of polynomials and exponential Diophantine equations, *Compositio Math.* **61** (1987), 135–157.

[3] B. Brindza, J.-H. Evertse and K. Győry, Bounds for the solutions of some diophantine equations in terms of discriminants, *J. Austral Math. Soc.* **51** (1991), 8–26.

[4] B. Brindza, K. Győry and R. Tijdeman, The Fermat equation with polynomial values as base variables, *Invent. Math.* **80** (1985), 139–151.

[5] Y. Bugeaud, Sur la distance entre deux puissances pures, *C. R. Acad. Sci. Paris* **322** (1996), 1119–1121.

[6] Y. Bugeaud and K. Győry, Bounds for the solutions of unit equations, *Acta Arith.* **74** (1996), 67–80.

[7] Y. Bugeaud and K. Győry, Bounds for the solutions of Thue–Mahler equations and norm form equations, *Acta Arith.* **74** (1996), 273–292.

[8] Y. Bugeaud and K. Győry, On binomial Thue–Mahler equations, *Periodica Math. Hungar.* (*to appear*).

[9] E. Friedman, Analytic formulas for regulators of number fields, *Invent. Math.* **98** (1989), 599–622.

[10] K. Győry, On discriminants and indices of integers of an algebraic number field, *J. Reine Angew. Math.* **324** (1981), 114–126.

[11] K. Győry, Bounds for the solution of decomposable form equations, *Publ. Math. Debrecen* **52** (1998), 1–31.

[12] K. Győry, Polynomials and binary forms with given discriminant (*to appear*).

[13] K. Győry and Kunrui Yu, Bounds for the solutions of some diophantine equations (*to appear*).

[14] J. Haristoy, Équations diophantiennes exponentielles, Thèse de docteur, *Strasbourg*, 2003.

[15] H. W. Lenstra, Jr., Algorithms in algebraic number theory, *Bull. Amer. Math. Soc.* **26** (1992), 211–244.

[16] E. M. Matveev, An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers II, *Izvestiya: Mathematics* **64** (2000), 1217–1269.

[17] I. Pink, On the differences between polynomial values and perfect powers, *Publ. Math. Debrecen* **63** (2003), 461–472.

[18] J. B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois J. Math.* **6** (1962), 64–94.

[19] A. Schinzel and R. Tijdeman, On the equation $y^m = P(x)$, *Acta Arith.* **31** (1976), 199–204.

[20] T. N. Shorey, A. J. van der Poorten, R. Tijdeman and A. Schinzel, Applications of the Gel'fond–Baker method to Diophantine equations, in: Transcendence Theory: Advances and Applications, *Academic Press*, *London – New York, San Francisco*, 1977, 59–77.

[21] T. N. Shorey and R. Tijdeman, Exponential Diophantine Equations, *Cambridge University Press*, *Cambridge*, 1986.

[22] J. Turk, On the difference between perfect powers, *Acta Arith.* **45** (1986), 289–307.

[23] M. Waldschmidt, Diophantine approximation on linear algebraic groups, *Springer*, 2000.

[24] Kunrui Yu, Linear forms in p-adic logarithms, *Acta Arith.* **53** (1989), 107–186.

[25] Kunrui Yu, p-adic logarithmic forms and group varieties III (*to appear*).

[26] Kunrui Yu and Ling-kei Hung, On binary recurrence sequences, *Indag. Math.* **6** (1995), 341–354.

[27] K. Zsigmondy, Zur Theorie der Potenzreste, *Monatsh. Math.* **3** (1892), 265–284.

kálmán győry
institute of mathematics
number theory research group of the hungarian academy of sciences
university of debrecen
4010 debrecen, p.o. box 12
hungary

E-mail: gyory@math.klte.hu

istván pink
university of debrecen
institute of mathematics
4010 debrecen, p.o. box 12
hungary

E-mail: pinki@math.klte.hu

ákos pintér
institute of mathematics
number theory research group of the hungarian academy of sciences
university of debrecen
4010 debrecen, p.o. box 12
hungary

E-mail: apinter@math.klte.hu

(Received August 18, 2004)